Heterogeneous agents in the Macroeconomy: Reduced-heterogeneity representations

Xavier Ragot

February 28, 2017

PRELIMINARY

Abstract

This chapter surveys the heterogenous agent models, which can be solved with rational expectations and perturbation methods. It focuses on the class of models which deliver a finite number of heterogenous agents as an equilibrium outcomes. Recent contributions show that many of the additional tools and techniques developed in the DSGE literature with a representative agent can easily be imported in this class of models, what considerably increases their empirical relevance. No-trade, small-heterogeneity and truncation methods are presented. Derivation of optimal policies is presented in these environments. Finally, the chapter discusses the relation with other heterogenous agents models which don't rely on rational expectations, namely agent-based models.

1 Introduction

Heterogeneity is now everywhere in the macroeconomy. Both on the normative and on the positive side, considering redistribution across agents with different wealth or economic behavior is obviously key for economic analysis. Economists mostly use the term "heterogeneity" to refer to the multiple dimensions according to which economic agents could differ. The public debate is mostly concerned by "inequalities" which refer to differences in income, wealth or consumption. Inequality should thus be understood as a subset of the broadest concept of heterogeneity for which a simple cardinal ranking of agents is possible (along the wealth dimension for instance). This being said, two lines of research coexist since many years, dealing with agents heterogeneity.

A first line of research assumes that agents are rational, such that their differences come either from characteristic they have before starting economic activities or from different histories of "shocks" they face in their life. The notion of shocks should be broadly understood, including income shocks, but also health shock, "family" shock (See Heathcote, Storesletten, and Violante (2009) for a discussion of sources of risk). The key tool to model agents heterogeneity is the class of models with uninsurable idiosyncratic risks, which have different names in the literature: They are called either the Bewley-Hugget-Imohoroglu-Aiyagari models, or the Standard Incomplete Market model (SIM), or even simply Heterogeneous-Agent models. This source of heterogeneity can be mixed with the introduction of the age dimension, in overlapping generation models to obtain a very rich representation of heterogeneity across households (Rios-Rull (1995) and Rios-Rull (1997) for an early survey). After the contribution of Krusell and Smith (1998) these models are solved with aggregate shocks (See Algan, Allais, DenHaan, Rendahl (2014) for a comparison of numerical methods).

Recent research now introduces many relevant frictions in this class of model, which were originally developed in the Dynamic Stochastic General Equilibrium (DSGE) literature with a representative agent. This frictions are sticky-prices, search-and-matching frictions on the labor market, habit-formation or limited-participation in financial markets (see Krusell, Mukoyama, Sahin (2010); Gornemann, Kuester, Nakajima (2012); Ravn and Sterck (2013); Kaplan, Moll Violante (2016); Challe, Matheron, Ragot and Rubio-Ramirez (2016) among others). These recent contributions have shown that heterogeneity is important for macroeconomists for positive and not only normative analysis. The effect of technology shocks, of fiscal or monetary policy are different between representative-agent world, and models where agents face uninsurable risks. To give a concrete example, Challe, Matheron, Ragot and Rubio-Ramirez (2016) show agents facing an expected increase in unemployment save to self-insure, as they are afraid to fall into unemployment. This contributes to a fall in aggregate demand, what reduces the incentives to post vacancies and increases unemployment. This negative feedback loop is a form of a "paradox of thrift" which is absent in representative agent models. This may have explained a third of the fall of the consumption of non-durable goods with respect to trends after 2008. Krueger, Mitman and Perri (2015) present other evidences of the importance of heterogeneity/inequality among households in the subprime crisis.

The goal of this chapter is to review recent methods to solve these models, which allow for an easy introduction of such frictions in general equilibrium. They are based on a simplification of the structure of heterogeneity (motivating the title of this chapter) and on simple perturbation methods. The models generate a finite number of equations to describe agents' heterogeneity. This endogenous outcome allows the use of econometric technics, such as estimation of the model with Bayesian tools. Moreover, one can derive normative implications from optimal policies in these environments.

The tools used in this chapter are designed to solve models with rational expectations (in a broad sense). These models thus differ from a second line of research on heterogeneous agents, which depart from rational expectations. The models are labelled Agents-based Models (ABM) and are now used in a vast literature, surveyed in many chapters of this Handbook. These models either assume that agents follow simple rules, and change rules according to interactions, or that they use expectations formation rules different from rational expectations (See Hommes (2006) for a survey). These second lines of research insist on the heterogeneity on expectation of the same variable of agents having similar information set to motivate their assumptions. The frontier between these two lines of research (rational and non-rational expectations) is still visible in the reference lists of various papers, but it can be expected to disappear progressively, as a growing literature investigate relevant models of expectation formation, which could be introduced in heterogeneous agent models. This would allow to empirically discipline the model along many dimensions, using information about inequality across households, about time-series properties and about expectations heterogeneity. Section 8 is dedicated to the discussion of the possible connections between models with and without rational expectations.

This chapter is mostly methodological. It details benchmark models generating reduced heterogeneity and sketch algorithm to solve them. Other approaches to solve heterogeneous-agent models with perturbation methods are used in the literature. The discussion and comparison with these alternative approaches is left for Section 7.

The presentation of this chapter follows the order of the complexity of the models. First, the basic problem is presented in Section 2 to lay down notations. Then, some economic problems can be investigated quantitatively in environments where agents don't trade in equilibrium. These no-trade equilibria are presented in Section 3. No-trade is too strong an assumption for models where the endogeneity of the amount of insurance (or self-insurance) is key for the economic problems under investigation. Section 4 presents a alternative class of models with small-heterogeneity models where heterogeneity is only preserved only for a subgroup of agents. Section 5 presents a general approach to reduce heterogeneity in incomplete insurance market models. In a nutshell, this theory is based on truncations of idiosyncratic histories, which endogenously delivers a finite (but arbitrarily large) number of different agents. Section 6 discuss the derivation of optimal policies in these environments. Section 7 compares the reduced-heterogeneity approach of this chapter with other methods using perturbation methods. Section 8 discusses the possible use of reduced-heterogeneity approaches for models not using rational expectations, such as Agent-Based Models. Section 9 is the conclusion. Empirical strategies to discipline and discriminate among the general class of heterogeneous agent models are discussed.

2 The economic problem and notations

2.1 The model

Time is discrete, indexed by $t \geq 0$. The aggregate risk is represented[1] by state variables $h_t \in \mathbb{R}^N$ in each period t. Typically, s_t can be the level of technology, the amount of public spending, and so on. It is assumed to be N-dimensionnal for the sake of generality. Key to the methods describe below is the fact that h_t is continuous to allow for perturbation methods. We will indeed solve for small variations in h_t or, in other words, for small changes in the aggregate state of the world. The idea is the same as linearizing a model around a well-defined steady-state for a representative agent model. It is always possible to take higher-order approximation, but usually a first-order approximation (linearizing the model) is enough to obtain key insights. The history of aggregate shocks up to period t is denoted $h^t = \{h_0, ..., h_t\}$.

Agents' problem

The specificity of heterogeneous agents models is that, on top of aggregate risk, each agent face uninsurable idiosyncratic risk, such that they will differ as time goes by, according to the realization of their idiosyncratic risk. More formally, assume that there is a continuum of length 1 of agents indexed by i.

[1]More formally, the aggregate risk is represented by a probability space $(\mathcal{S}^\infty, \mathcal{F}, \mathbb{P})$. See Heathcote (2005) or Miao (2006) for a more formal presentation.

Agents face time-varying idiosyncratic risk. At the beginning of each period, agents face an idiosyncratic labor productivity shock $e_t \in \mathcal{E} \equiv \{e_1, .., e_E\}$ that follows a discrete first-order Markov process with transition matrix $M(h_t)$, which is a $E \times E$ Markov matrix. The probability $M_{e,e'}(h_t)$, $e, e' \in \mathcal{E}$ is the probability for an agent to switch from individual state $e_t = e$ at date t to state $e_{t+1} = e'$ at date $t+1$, when aggregate state is he in period t. At period t, $e^t = \{e_0, ..., e_t\} \in \mathcal{E}^{(t+1)}$ denotes an history of the realization of idiosyncratic shock, up to time t. The fact that the idiosyncratic state-space is discrete is crucial for the methods presented below, but it not restrictive for the application found in the literature. The idiosyncratic states considered are often employment-unemployment or two states endowment economy (as in Huggett (1993)), or different idiosyncratic productivity levels to match the empirical process of labor income (Heathcote (2005) use a 3-state process, Aiyagari (1994) use 7-states process). More generally, any continuous first-order process can be approximated by a discrete process, using the Tauchen (1986) procedure.

In what follows, and without loss of generality, I will consider a two-state process where agents can be either employed, when $e_t = 1$ or unemployed, when $e_t = 0$. In this latter case, the agent receives must supply a quantity of labor δ for home production to obtain a quantity of goods δ: The labor choice is constrained. The probability to stay employed is denoted $\alpha_t \equiv M_{1,1}(h_t)$ thus $1 - \alpha_t$ is the job-separation rate. The probability to stay unemployed is denoted as $\rho_t \equiv M_{0,0}(h_t)$, such that $1 - \rho_t$ is the job finding rate in period t.

Agents have a discount factor β and a period utility function $U(c, l)$, which is increasing in consumption c and decreasing in labour supply l. In addition U is twice-differentiable and has standard concavity properties.

Market structure. Agents can't buy assets contingent on their next-period employment status (otherwise, they could buy some insurance), but can only save in a asset, whose return only depends on the history of the aggregate states h^t.

The typical problem of an agent facing incomplete insurance-market is the following

$$\max_{\left(a_t^i(.,.),c_t^i(.,.)\right)_{t\geq 0}} \mathbb{E}_0 \quad \sum_{t=0}^{\infty} \beta^t \quad U\left(c_t^i, l_t^i\right) \tag{1}$$

$$a_{t+1}^i + c_t^i \quad = \quad e_t^i w_t l_t^i + (1-e_t^i)\delta + (1+r_t)a_t^i, \text{ for all } e^N \in \mathcal{E}^N, \tag{2}$$

$$c_t^i, l_t^i \geq 0, \qquad a_t^i \geq -\bar{a}, \text{ for all } e^N \in \mathcal{E}^N, \tag{3}$$

$$a_0^i \quad \text{are given,} \tag{4}$$

where w_t is the wage rate in period t and r_t is the return on saving between period $t-1$ and period t. a_{t+1}^i is the saving of agent i in period t, and c_t^i, l_t^i are respectively consumption and labor supply of agent i in period t. More rigorously, aggregate variables should be understood as a function of the history of aggregate shock h^t, thus as $r(h^t)$ and $w(h^t)$, whereas idiosyncratic variables should be understood as functions of both aggregate and idiosyncratic histories , as $a_{t+1}^i(e^{i,t}, h^t)$ for instance. The decisions in each periods are subject to the non-negativity constraints (3). Importantly, agents can't borrow more than the amount $\bar{a}$ in each period.

Production

Markets are competitive and a representative firm produces with capital and labor. The production function is $Y_t = A_t K_t^{\lambda} L_t^{1-\lambda} + (1-\mu)K_t$,where μ is the capital depreciation rate, and A_t is the technology level, which is affected by technology shocks. The first-order conditions of the firm imply that factor prices are

$$r_t + \mu = \lambda A_t K_t^{\lambda-1} L_t^{1-\lambda}$$

and

$$w_t = (1-\lambda) A_t K_t^{\lambda} L_t^{-\lambda}$$

where K_t is the aggregate capital stock and L_t is aggregate labor supply.

The technology shocks is defined as the standard AR(1) process $A_t \equiv e^{a_t}$, with

$$a_t = \rho^a a_{t-1} + \epsilon_t^a$$

with $\epsilon_t^a \sim \mathcal{N}\left(0, (\sigma^a)^2\right)$.

2.2 Equilibrium definition and intuition to reduce the state space

We can provide the equilibrium definition and the main idea to reduce the state space. First, in the general case, as time goes by there is an increasing number of different agents, due to the heterogeneity in idiosyncratic histories. Instead of thinking in sequential terms (i.e.. following the history of each agent from period 0 to any period t), Huggett (1993) and Aiyagari (1994) have shown that the problem can be stated in recursive form, if ones introduces a continuous distribution as a state variable, when there is no aggregate shocks[2].

Indeed, define as $F_t : [-\bar{a}; +\infty) \times \{0,1\} \to \mathbb{R}^+$ the cross-sectional cumulative distribution over capital holdings and idiosyncratic state in period t. In the general case, an equilibrium of this economy is 1) a policy rule for each agent solving its individual maximization problem 2) factor prices which are consistent with the firm first order condition 3) financial and labor markets clear for each period $t \geq 0$:

$$\int a_{t+1}(a,e)dF_t(a,e) = K_{t+1} \tag{5}$$

where $a_{t+1}(a,e)$ is the saving in period t of an household having initial wealth a and being in state $e \in \{0,1\}$, and

$$\int l_t(a,e)dF_t(a,e) = L_t \tag{6}$$

and finally 4) a low of motion for the cross-sectional distribution F_t which is consistent with agents' decision rule at each date. This law of motion can be written as (Following the notation of Algan et al. (2014))

$$F_{t+1} = \Upsilon(h_{t+1}, h_t, F_t)$$

The literature on heterogeneous-agent models has tried to find solution techniques to approximate the very complex object Υ, which maps distribution and shocks into distributions.

The basic idea. The basic idea to reduce the state space is first to go back to the sequential representation. If at any period t, only the last N periods are necessary to know the wealth of any agents, then only the truncated history $e^{N,t} = \{e_{t+1-N}, ..., e_t\} \in \mathcal{E}^N$ is necessary to "follow" the whole distribution of agents, in a sense made clear below. In this economy, there are only

[2]The existence of a recursive equilibrium with aggregate shocks is still an open question, See Miao (2006) for a discussion. This difficulty will not exist for the class of equilibria presented in this chapter.

2^N different agents at each period, instead of a continuous distribution. This number can be large but it is finite and all standard perturbation techniques can be applied.

The next Section presents equilibria found in the literature. The first one is no-trade equilibrium where $N = 1$, the second one ireduced-heterogeneity equilibrium and the last one is the general case for arbitrary N.

3 No-trade equilibria

3.1 No-trade equilibria in the Bewley environments

A first simple way to generate a tractable model is to consider environments which endogenously generates no-trade equilibria with transitory shocks. This class of equilibrium was introduced by Krusell, Mukoyama and Smith (2011) to study asset prices with time-varying idiosyncratic risk. recent developments show that they can be useful for macroeconomic analysis. These equilibria are based on two assumptions.

First, assets are in zero net supply and production only necessitates labor ($\lambda = 0$ in the production function) . The first consequence, is that the total amount of saving must be equal to the total amount of borrowing *among* households. The second consequence is that the real wage is only the technology level in each period $w_t = A_t$. Second, it is assumed that borrowing limit is 0, $\bar{a} = 0$. As a direct consequence, as agents can't borrow, there is no assets in which agents can save: $a_t^i = 0$ for all agents i in any period t. Those equilibria are not interesting for generating a realistic cross-section of wealth, but they can be interesting to investigate the behavior of the economy in front of time-varying uninsurable risk. Indeed, the price of any asset is determined by the highest price than any agent is willing to pay.

Denoting $\{1\}$ the employed agents and $\{0\}$ the unemployed agents, one can now state the problem recursively. The value functions for employed and unemployed agents are

$$\begin{aligned} V(a,h,\{1\}) &= \max{}_{c,l,a'} U(c,l) + \beta \mathbb{E} \left(\alpha' V(a',h',\{1\}) + (1-\alpha')V(a',h',\{0\}) \right) \\ & \qquad a' + c = Al + a(1+r) \\ & \qquad a' \geq 0 \end{aligned}$$

and

$$\begin{aligned} V(a,h,\{0\}) &= \max{}_{c,a'} U(c,\delta) + \beta \mathbb{E}\left(\rho' V(a',h',\{0\}) + (1-\rho')V(a',h',\{1\})\right) \\ &\qquad a' + c = \delta + a(1+r) \\ &\qquad a' \geq 0 \end{aligned}$$

where the expectation operator $\mathbb{E}$ is taken for the aggregate shock only.

As no-agent can save, we have $a = 0$ for all agents, one can thus see that all employed agents consume $c_1 = Al_1$ and supply the same quantity of labor l_1, whereas unemployed agent simply consume $c_0 = \delta$ and the labor supply is obviously given by $l_0 = \delta$.

The equilibrium can be derived using a guess-and-verify structure. Indeed, for general values of the parameters derived below, unemployed agents are credit constrained : they would like to borrow, and employed agents would like to save. As a consequence, they are the marginal buyer of the asset (although in zero-net supply) and make the price.

Deriving the first-order conditions of the previous program and *then* using these values, one finds :

$$\begin{aligned} AU_c(c_1,l_1) &= -U_l(c_1,l_1) \\ U_c(c_1,l_1) &= \beta E\left(1+r'\right)\left(\alpha' U_c(c_1',l_1') + (1-\alpha')U_c(c_0',0)\right) \end{aligned}$$

and the conditions for unemployed agents to be credit-constrained at the current interest rate is

$$U_c(c_0,0) > \beta E\left(1+r'\right)\left(\rho' U_c(c_0',0) + (1-\rho')U_c(c_1',l_1')\right)$$

Specification of the functional forms.

Assume that

$$\begin{cases} U(c,l) = & \frac{c^{1-\sigma}-1}{1-\sigma} - \chi \frac{l^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}} \text{ if } \sigma \neq 1 \\ U(c,l) = & log(c) - \chi \frac{l^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}} \text{ if } \sigma = 1 \end{cases}$$

σ is the curvature of the utility function (not directly equal to risk aversion to the endogenous labor supply), and ϕ is the Frisch elasticity of Labor supply, ranging from 0.3 to 2 in applied work (see Chetty et al. (2011) for a discussion). χ is a parameter scaling the supply of labor in steady state. With this specification one finds

$$\begin{aligned} A_t c_{1,t}^{-\sigma} &= \chi l_{1,t}^{\frac{1}{\phi}} \\ c_{1,t}^{-\sigma} &= \beta \mathbb{E}\left(1+r_{t+1}\right)\left(\alpha_{t+1} c_{1,t+1}^{-\sigma} + (1-\alpha_{t+1})\delta^{-\sigma}\right) \end{aligned} \tag{7}$$

and the conditions for unemployed agents to be credit-constrained at the current interest rate is

$$\delta^{-\sigma} > \beta \mathbb{E}\left(1+r_{t+1}\right)\left(\rho_{t+1}\delta^{-\sigma} + (1-\rho_{t+1}) c_{1,t+1}^{-\sigma}\right)$$

From the budget constraint of employed agents $c_{1,t} = A_t l_{1,t}$ and the labour choice in (7), we obtain

$$c_{1,t}^{1+\phi\sigma} = A_t^{1+\phi}/\chi^{\phi}$$

The technology process is the following

$$A_t = e^{a_t}$$

where a_t is an AR(1) process specified above. Three shocks hit the economy: a shock to technology level, ϵ_t^a, a shock to the probability to stay employed ϵ_t^α and a shock to the probability to stay unemployed ϵ_t^ρ . Those three shocks are white noise with standard deviation equal to $\sigma^a, \sigma^\alpha$and σ^ρ respectively, $\mathcal{N}\left(0,(\sigma^a)^2\right), \mathcal{N}\left(0,(\sigma^\alpha)^2\right), \mathcal{N}\left(0,(\sigma^\rho)^2\right)$. More formally,

$$\begin{bmatrix} a_t \\ \alpha_t - \bar{\alpha} \\ \rho_t - \bar{\rho} \end{bmatrix} = \begin{bmatrix} \rho^a & 0 & 0 \\ 0 & \rho^\alpha & 0 \\ 0 & 0 & \rho^\rho \end{bmatrix} \begin{bmatrix} a_{t-1} \\ \alpha_{t-1} - \bar{\alpha} \\ \rho_{t-1} - \bar{\rho} \end{bmatrix} + \begin{bmatrix} \epsilon_t^a \\ \epsilon_t^\alpha \\ \epsilon_t^\rho \end{bmatrix}$$

In the previous process, the covariation between the exogenous shocks are 0, but alternative specifications are easy to introduce. The steady-state value of α_t is $\bar{\alpha}$ and the steady-sate level of ρ_t is $\bar{\rho}$.

Steady state. To use perturbation method, we first solve for the steady sate and then consider first-order deviation from the steady-state. In steady state $A = 1$, and we get from the to equations in (7),

$$c_1 = \chi^{-\frac{\phi}{1+\sigma\phi}} \tag{8}$$

and

$$1+r=\frac{1}{\beta}\left(\bar{\alpha}+(1-\bar{\alpha})\left(\frac{\delta}{c_1}\right)^{-\sigma}\right)^{-1}$$

Putting some numbers allows estimating the order of magnitude. Consider the period to be a quarter. The previous equality shows that the effect of uninsurable risk on the interest rate is the the consumption inequality between employed and unemployed agents (irrespective of labor supply elasticity for instance). Chodorow-Reich and Karabarbounis (2014) estimate a decrease in consumption of non-durable goods of households falling into unemployed between 10% and 20%. As a consequence, one can take the conservative value $\frac{\delta}{c_1}=.9$. The quarterly job loss probability is roughly 5% (see Challe and Ragot (2014) for a discussion), and the discount factor is $\beta=.99$. One finds a real interest rate $r=0.45\%$ for $\sigma=1$, and $r=-0.002$ when $\sigma=2.$, instead of $r=1\%$ in the complete market case (where $\alpha=1$ for instance).

3.2 Preserving time-varying precautionary saving in linear model

The effect of time-varying precautionary saving is preserved in linear model for all the environment studied in this chapter. This is best understood in this simple framework. Indeed, linearizing the Euler equation in (7), one finds that

$$\hat{c}_{1,t}=\mu_1\mathbb{E}\hat{c}_{1,t+1}+\mu_2\mathbb{E}\tilde{\alpha}_{t+1}-\frac{1}{\sigma}\mathbb{E}\tilde{r}_{t+1} \tag{9}$$

where

$$\mu_1=\alpha\beta\left(1+r\right)\text{ and }\mu_2=\frac{\beta\left(1+r\right)}{\sigma}\left(\frac{\delta^{-\sigma}-c_1^{-\sigma}}{c_1^{-\sigma}}\right);$$

with the values given above, one finds $\mu_1=.94$ and $\mu_2=.1$, when $\sigma=1$. To give an order of magnitude, an increase in 10% in the expected job-separation rate (a decrease in α) has the same effect as an increase of 1% in the real interest rate. A second key implication is the value of $\mu_1<1$ in front of $\mathbb{E}\hat{c}_{1,t+1}$ this has dramatic implications for monetary policy compared to the complete market case, where there is no μ_1 coefficient. These implications are studied by McKay et al. (2016) in this type of environment.

One can see that probability to stay employed α_t has a first order effect on the consumption decision in (9). The reason for this result is that we are not linearizing around a risk-less steady state. We are linearizing around a steady state where idiosyncratic risk is preserved. As a consequence, there are two different marginal utilities that agents can experienced in the steady state : either $c_1^{-\sigma}$ if employed, or $\delta^{-\sigma}$ if unemployed. As a consequence, the term $\frac{\delta^{-\sigma}-c_1^{-\sigma}}{c_1^{-\sigma}}$ in μ_2 represents the lack of insurance in steady state. This term scales the reaction of consumption to changes in the idiosyncratic probability to change employment status.

Linearizing the labor-supply equation, one finds $\hat{c}_{1,t} = \frac{1+\phi}{1+\phi\sigma} a_t$. Plugging this expression in (9), one finds that the value of the interest rate is pinned down by the shocks (using $\mathbb{E}\tilde{\alpha}_{t+1} = \rho^{\alpha}\tilde{\alpha}_t$ and $\mathbb{E}a_{t+1} = \rho^a a_t$):

$$\mathbb{E}\tilde{r}_{t+1} = \sigma\mu_2\rho^{\alpha}\tilde{\alpha}_t - \sigma\frac{1+\phi}{1+\phi\sigma}\left(1-\mu_1\rho^a\right)a_t$$

These no-trade equilibria are extreme representation of market incompleteness, as the consumption levels are exogenous. They can nevertheless be useful in DSGE models. For instance, Ravn and Sterck (2013) use the same trick to study an incomplete-insurance market model where households can be either employed and unemployed. The simplification on the households side allows to enrich the production side and to consider sticky prices, introducing quadratic cost of price adjustment à la Rotemberg (1982), search-and-matching frictions on the labor market and downward nominal wage rigidities. In this environment, they consider two types of unemployed workers which differ in their search efficiency and therefore in their job finding probabilities. They use this model to account for change in the US labor market after the great recession. They focus in particular on the distinction of shifts in the Beveridge and of movements along the Beveridge curve. Challe (2017) uses a no-trade equilibrium to analyze optimal monetary policy with sticky prices on the goods market and sear-and-matching frictions on the labor market. He shows that optimal monetary policy reaction is more expansionary after a cost-push shocks when markets are incomplete (compared to the complete market environment), because there are additional gains to reduce unemployment when markets are incomplete.

3.3 No-trade equilibrium with permanent shocks

A second line of literature to generate tractable no-trade equilibria is based on the Constantinides and Duffie (1996) environment. These authors consider permanent idiosyncratic risk (instead of transitory risk as in the previous framework) and show that one can study market allocations and assets prices with no-trade. Recently, Heathcote, Storesletten and Violante (2014) generalize this framework to quantify risk sharing, and to decompose inequality into life-cycle shocks versus initial heterogeneity in preferences and productivity. Closed-form solutions are obtained for equilibrium allocations and for moments of the joint distribution of consumption, hours, and wages.

These no-trade equilibria are useful to provide a first quantification of new mechanisms generated by incomplete insurance markets. Nevertheless, they can't consider the macroeconomic effect of changes in savings after aggregate shocks. Small-theterogeneity models have been developed to consider this important additional channel in tractable environments.

4 Small-heterogeneity models

Small-heterogeneity models are classes of equilibria where agents do save but where the equilibrium distribution of wealth endogenously features a finite state space. Three classes of equilibria can be found in the literature. Each type of equilibrium has its own merit according to the question under scrutiny. I present the first one in detail, and the two other ones more rapidly, as the algorithms to solve for the equilibrium are very similar. The last class of equilibrium may be more suited for quantitative analysis as the conditions for the equilibrium to exist are easier to check.

4.1 Models based on assumptions about labor supply

4.1.1 Assumptions

The first class of equilibria is based on two assumptions.

1) First, it is assumed that agents choose their labor supply when employed and that the disutility of labor supply is linear. If c is consumption and l is labor supply, the period utility function is

$$U(c,l) = u(c) - l$$

The implication of this assumption is that the first order condition for labour supply pins down the marginal utility of consumption of employed agents. This assumption is used in Scheikman and Weiss (1986) and in Lagos and Wright (2005) to simplify heterogeneity in various environments.

2) The second assumption is that the credit constraint is tighter than the natural borrowing

limit

$$\bar{a} > -\delta/r \tag{10}$$

where r is the steady-state interest rate. This concept is introduced by Aiyagari (1994), and it is the loosest credit constraints, which ensure that consumption is positive. The implication of this assumption is that unemployed agents will hit the credit constraint after a finite periods of unemployment. This property is key to reduce the state space and we discuss it below.

4.1.2 Equilibrium structure

To simplify the exposition, the equilibrium is presented using a guess-and-verify strategy. For the sake of clarity, the time index is kept to variables (although not necessary in the recursive exposition). Assume that all employed agents consume and save the same amount in each periods t, $c_{0,t}$ and $a_{0,t+1}$ respectively. In addition assume that all agents unemployed for k periods consume and save the same amount, denoted, $c_{k,t}$ and $a_{k,t+1}$, for $k \geq 1$ respectively. In addition, assume that agents unemployed for L periods are credit constrained, and that this number is not time-varying ($L$is an equilibrium object). This last assumption is important and will be justified below.

Denote as $V_k(a_t, X_t)$ the value function of agents in state $k = 0, 1, ...$ (0 is employed agents, here), where X_t is the set of variables specified below which are necessary to form rational expectations[3].

We have for employed people

$$V_0(a, X_t) = \max_{c_{0,t}, a_{0,t+1} l_t} u(c_{0,t}) - l_t + \beta \mathbb{E} \left(\alpha_{t+1} V_0(a_{0,t+1}, X_{t+1}) + (1 - \alpha_{t+1}) V_1(a_{0,t+1}, X_{t+1}) \right)$$
$$a_{0,t+1} + c_{0,t} = w_t l_t + a(1 + r_t)$$
$$a_{0,t+1} \geq -\bar{a}$$

and for all unemployed people, $k \geq 1$

$$V_k(a, X_t) = \max u(c_t) - \delta + \beta \mathbb{E} \left(\rho_{t+1} V_0(a_{k,t+1}, X_{t+1}) + (1 - \rho_{t+1}) V_{k+1}(a_{k,t+1}, X_{t+1}) \right)$$
$$a_{k,t+1} + c_{k,t} = \delta + a(1 + r_t)$$
$$a_{k,t+1} \geq -\bar{a}$$

[3]We introduce the tim subscript in the recursive formulation to ease the understanding of the timing of the model.

As credit constraints bind for agents unemployed for $k \geq L$ periods, we have for these agents $a_{k,t+1} = -\bar{a}$.

We can derive the set of first order conditions. For employed agents;

$$\begin{aligned} u'(c_{0,t}) &= 1/w_t \\ u'(c_{0,t}) &= \beta \mathbb{E}(1+r_{t+1}) \left(\alpha_{t+1} u'(c_{0,t+1}) + (1-\alpha_{t+1}) u'(c_{1,t+1})\right) \end{aligned}$$

For unemployed agents, for $k = 1...L-1$.

$$u'(c_{k,t}) = \beta \mathbb{E}(1+r_{t+1}) \left(\rho_{t+1} u'(c_{k+1,t+1}) + (1-\rho_{t+1}) u'(c_{0,t+1})\right)$$

Note that when $L = 1$, such that the credit constraints bind after one period of unemployment, then the previou equations don't exist. This case is studied more precisely below.

The conditions define a system of $2(L+1)$ equations.

$$\begin{aligned} 1/w_t &= \beta \mathbb{E}(1+r_{t+1}) \left(\alpha_{t+1}/w_{t+1} + (1-\alpha_{t+1}) u'(c_{1,t+1})\right), \\ u'(c_{k,t}) &= \beta \mathbb{E}(1+r_{t+1}) \left(\rho_{t+1} u'(c_{k+1,t+1}) + (1-\rho_{t+1})/w_{t+1}\right), \quad \text{for } k = 1..L-1 \\ a_{k,t+1} + c_{k,t} &= \delta + a_{k-1,t}(1+r_t), \qquad \text{for } k = 1..L \\ u'(c_{0,t}) &= 1/w_t \\ a_{L,t} &= -\bar{a} \end{aligned}$$

These equations form a system in the $2(L+1)$ variables $(c_{k,t}, a_{k,t})_{k=0..L}$. These equations confirm the intuition that all employed agents consume and same the same amount. How is it possible? This comes from the labor choice, which provides some insurance. Indeed, as soon as an unemployed agent for k period in period $t-1$ becomes employed in period t, then it works the necessary amount, denoted as $l_{k0,t}$ to consume $c_{0,t}$. This amount is given by the budget constraint of employed households

$$\begin{aligned} l_{k0,t} &= \left(a_{0,t+1} + c_{0,t} - a_{k,t}(1+r_t)\right)/w_t, \quad k = 0,..,L-1 \\ l_{k0,t} &= \left(a_{0,t+1} + c_{0,t} + \bar{a}(1+r_t)\right)/w_t, \quad k = L,..,\infty \end{aligned} \tag{11}$$

(The previous equation is indeed valid for $k = 0$). Finally, note that for agents $k \geq L+1$, we simply have, from the budget constraint

$$c_{k,t} = \delta - r_t \bar{a}$$

Thanks to the assumption about the credit constraint given by (3) and the assumption of small aggregate shocks (to use perturbation methods), this amount will be positive.

This almost concludes the description of the agents decision. The last step is to follow the number of employed and of each type of unemployed agents. Denote as $n_{k,t}$ the number of agents in state $k = 0, .., L$ in each period t. We have the law of motion of each type of agents

$$\begin{aligned} n_{0,t} &= \alpha_t n_{0,t-1} + (1-\rho_t)(1-n_{0,t-1}) \\ n_{k,t} &= \rho_t n_{k-1,t-1}, \qquad \text{for } k = 1, ..., \infty \end{aligned}$$

The first equation states that the number of employed agents is equal to the number of employed agents who keep their job (first) term, plus the number of unemployed agents which is $1 - n_{0,t-1}$, who find a job. The second equation states that the number of agents unemployed for k periods at date t, are the number of agents unemployed for $k-1$ periods at the previous date, who stay unemployed.

The number of $k0$ agents (i.e. employed agents at date t, who were unemployed for k periods at date $t-1$) is

$$n_{k0,t} = (1-\rho_t) n_{k,t-1}$$

The number of credit constrained agents is denoted as n_t^c and it is simply

$$n_t^c = 1 - \sum_{k=0}^{L-1} n_{k,t}$$

In this equilibrium, the capital market equilibrium is simply

$$K_t = \sum_{k=0}^{L-1} n_{k,t} a_{k,t} - n_t^c \bar{a}$$

and

$$L_t = \sum_{k=0}^{L-1} n_{k0,t} l_{k0,t} + (1-\rho_t)\, n_{t-1}^c l_{L0,t}$$

here we used the fact that all constrained agents work the same amount when they find a job, due to condition (11).

Due to this assumption, the marginal utility of all employed agents is $u'(c_t) = 1/w_t$ in all periods. As a consequence, this marginal utility does not depend on the history of agents on the labor market, what considerably simplifies the equilibrium structure.

4.1.3 The system

We can now present the whole system of equations :

$$
\begin{aligned}
1/w_t &= \beta\mathbb{E}(1+r_{t+1})\left(\alpha_{t+1}/w_{t+1} + (1-\alpha_{t+1})u'(c_{1,t+1})\right), && \\
u'(c_{k,t}) &= \beta\mathbb{E}(1+r_{t+1})\left(\rho_{t+1}u'(c_{k+1,t+1}) + (1-\rho_{t+1})/w_{t+1}\right), && \text{for } k = 1,...,L-1 \\
a_{k,t+1} + c_{k,t} &= \delta + a_{k-1,t}(1+r_t), && \text{for } k = 1,...,L \\
u'(c_{0,t}) &= 1/w_t && \\
a_{L,t} &= -\bar{a} && \\
l_{k0,t} &= \left(a_{0,t+1} + c_{0,t} - a_{k,t}(1+r_t)\right)/w_t, && \text{for } k = 0,..,L-1 \\
l_{L0,t} &= \left(a_{0,t+1} + c_{0,t} + \bar{a}(1+r_t)\right)/w_t, && \\
n_{0,t} &= \alpha_t n_{0,t-1} + (1-\rho_t)(1-n_{0,t-1}) && \\
n_{1,t} &= (1-\alpha_t)n_{0,t} && \\
n_{k,t} &= \rho_t n_{k-1,t-1}, && \text{for } k = 2,...,L-1 \\
n_{00,t} &= \alpha_t n_{0,t} && \\
n_{k0,t} &= (1-\rho_t)n_{k,t-1}, && \text{for } k = 1,...,L-1 \\
n_t^c &= 1 - \sum_{k=0}^{L-1} n_{k,t} && \\
K_t &= \sum_{k=0}^{L-1} n_{k,t}a_{k,t} - n_t^c\bar{a} && \\
L_t &= \sum_{k=0}^{L-1} n_{k0,t}l_{k0,t} + (1-\rho_t)\, n_{t-1}^c l_{L0,t} && \\
r_t &= \lambda A_t K_t^{\lambda-1} L_t^{1-\lambda} - \mu && \\
w_t &= (1-\lambda)A_t K_t^{\lambda} L_t^{-\lambda} &&
\end{aligned}
$$

This system is large but finite. There are $5L+8$ equations for $5L+8$ variables

$((c_{k,t}, a_{k,t+1}, l_{k0,t})_{k=0,..,L}, (n_{k,t}, n_{k0,t})_{k=0,..,L-1}, n_t^c, K_t, L_t, r_t, w_t)_{t=0...\infty}$. For any process for the exogenous shocks (α_t, ρ_t, A_t) one may think that it is possible to simulate this model. This is not the case because one key variable is not determined: L.

4.1.4 Algorithm : Finding the value of L

The value of L can be found using steady-state solution of the previous system. This idea is to find the steady-state for any L and iterate over L to be find the value for which L agents are credit constrained, whereas $L-1$ agents are not. The algorithm to find L and the steady is the following (finding the steady state is not difficult because the problem is block-separable). I simply drop the time subscript to denote steady-state values.

Algorithm

1. Take $L \geq 1$ as given
    (a) take r as given
        i. From r deduce w using the FOCs of the firms
        ii. Solve for the consumption of agents $c_{k=0...L}$ using the Euler equations of the agents, from $k=1$ to $k=L$.
        iii. Solve for the saving of the agents from a_L down to a_0 using the budget constraint of all agents, and the values $c_{k=0...L}$.
        iv. Solve for the labor supply of employed agents l_{k0} $k=0,...,L$
        v. Solve for the share of agents n_k, n_{k0} for $k=0,...,L-1$ and n^c.
        vi. Find the aggregate capital stock K
    (b) Iterate of over r, until the financial market clears, i.e. until $r = \lambda K^{\lambda-1} L^{1-\lambda} - \mu$
2. Iterate over L, until

$$\begin{aligned} a_{L-1} &> -\bar{a} \\ u'(c_L) &> \beta(1+r)\left((1-\bar{\rho})\,u'(c_0) + \rho u'(c_{L+1})\right) \end{aligned} \tag{12}$$

where $c_{L+1} = \delta - r\bar{a}$.

4.1.5 Simulations

Once the steady-state value of L and the steady-state value of the variables are determined, one can simulate the model using standard perturbation methods. One can use DYNARE to simulate first and second approximation of the model, compute second moments and so on. The distribution of wealth is summarized by the vector $(n_{k,t}, a_{k,t})_{k=0,..,L-1}$ and belong to the state space of agents X_t to form rational expectations. In these simulations, one has to check that the aggregate shock is small enough such that L is indeed constant over time. One must thus check that the condition (12) is satisfied not only in the steady state but also during the simulations.

4.1.6 References and limits

In the Bewley-Huggett-Aiyagari environment, Algan, Challe and Ragot (2011) use to this framework to investigate the impact of money injections in a model where agents use money to self-insure against idiosyncratic shocks. Challe, Le Grand and Ragot (2013) use this assumption to study the effect of an increase in public debt on the yields curve in a environment where agents use safe assets of various maturities to self-insurance against idiosyncratic risk. They show that an increase in idiosyncratic risk decreases both the level and the slope of the yield curve. In addition an increase in public debt increases both the level and the slope of the yield curve. Le Grand and Ragot (2016a) use this assumption to consider insurance for aggregate risk in this environments. Introducing derivative assets, such as options, in an environment where agents use a risky asset to self-insure against idiosyncratic risk, they show that the time-variations in the volume of traded derivative assets are consistent with empirical findings.

This framework is interesting to investigate the properties of time-varying precautionary saving in finance (for instance to study asset prices) but it is not well suited for the macroeconomy. Indeed, the elasticity of labor supply is much too high compared to empirical finding (the Frisch elasticity is here infinite, whereas it is between 0.3 and 1 in the data, see Chetty et al. (2011) for a discussion). In addition, all employed agents consume the same amount, which is pinned down by the real wage, which obviously a counterfactual. For this reason, other frameworks with positive trade have been developed.

4.2 Models based on linearity in the period-utility function

Challe and Ragot (2014) present an alternative environment, consistent with any value of the elasticity of labor supply. This framework can thus be used in macroeconomic environments to model time-varying movements in inequality. We describe the empirical relevance and the modeling strategy in Section 4.2.3 after the presentation of the model.

4.2.1 Assumptions

The model relies on three assumptions.

1) First, instead on introducing linearity in the labor supply, the linearity is in the utility of consumption. More precisely, it is assumed that there exists a threshold c^*such that the period utility function is strictly concave for $c < c^*$ and linear for $c \geq c^*$.

2) Second, the borrowing limit is assumed to be strictly higher than the natural borrowing limit, as before.

3) Third it is assumed that the discount factor of agents is such that all employed agents consume an amount $c_t > c^*$and all unemployed agents consume an amount $c_t < c^*$.

The linear-after-a-threshold utility function was introduced by Fishburn (1977) in decision theory to model differently behavior in front of gains and losses.

The period utility function is thus

$$\tilde{u}'(c) = \begin{cases} u'(c) & \text{if } c < c^*, \\ \eta & \text{if } c^* \leq c, \end{cases} \tag{13}$$

where the function $u(.)$ is increasing and concave. The slope of the utility function must be low enough to obtain global concavity, that is $u'(c^*) > \eta$.

4.2.2 Equilibrium Structure

To save some space, we now focus on the households' program. Assume that labor is inelastic as a useful benchmark (introducing elastic labor is very simple in this environment). All employed agents supply one unit of labor.

We have for employed people

$$V_0(a, X_t) = \max_{c_{0,t}, a_{0,t+1} l_t} \tilde{u}(c_{0,t}) - 1 + \beta \mathbb{E} \left(\alpha_{t+1} V_0(a_{0,t+1}, X_{t+1}) + (1 - \alpha_{t+1}) V_1(a_{0,t+1}, X_{t+1}) \right)$$
$$a_{0,t+1} + c_{0,t} = w_t + a(1 + r_t)$$
$$a_{0,t+1} \geq -\bar{a}$$

and for all unemployed people, $k \geq 1$

$$V_k(a, X_t) = \max \tilde{u}(c_t) - \delta + \beta \mathbb{E} \left(\rho_{t+1} V_0(a_{k,t+1}, X_{t+1}) + (1 - \rho_{t+1}) V_{k+1}(a_{k,t+1}, X_{t+1}) \right)$$
$$a_{k,t+1} + c_{k,t} = \delta + a(1 + r_t)$$
$$a_{k,t+1} \geq -\bar{a}$$

One can solve for the order conditions following the same steps as before. Using the same notations as in the previous Section, denote as k agents, the agents unemployed for k periods. Assuming that credit constraints are binding after L periods of unemployment, one can find consumption and saving choices.

The key difference between this environment and the one presented in the previous Section, is that employed agents will not differ according to their labor supply (which is inelastic), but by their consumption level. Denote as $c_{k0,t}$ the consumption at date t of employed agents who were unemployed for k periods at date $t - 1$, and denote (as before) as $c_{k,t}$ (for $k = 1, ..., L$) the consumption of unemployed agents who are unemployed for k periods at date t. The households are now described by the the vector $(a_{k,t}, c_{k0,t})_{k=0,...,L}$ and $(c_{k,t})_{k=1,...,L}$ solving

$$\begin{array}{ccccc}
\eta & = & \beta \mathbb{E}(1 + r_{t+1}) \left(\alpha_{t+1} \eta + (1 - \alpha_{t+1}) u'(c_{1,t+1}) \right), & \\
u'(c_{k,t}) & = & \beta \mathbb{E}(1 + r_{t+1}) \left(\rho_{t+1} u'(c_{k+1,t+1}) + (1 - \rho_{t+1}) \eta \right), & \text{for } k = 1, ..., L - 1 \\
a_{k,t+1} + c_{k,t} & = & \delta + a_{k-1,t}(1 + r_t), & \text{for } k = 1, ..., L \\
a_{0,t+1} + c_{k0,t} & = & w_t + a_{k,t}(1 + r_t), & \text{for } k = 0, ..., L \\
a_{L,t} & = & -\bar{a} &
\end{array}$$

One can check that this is a system of $3L + 2$ equations for $3L + 2$ variables. The number of each type of agents can be followed as in the previous Section.

One may find this environment more appealing than the one in the previous Section, as it does not rely on an unrealistic elasticity of labor supply. The problem is nevertheless that there are additional conditions for the equilibrium existence which limit the use of such a framework. Indeed, one has first to solve for steady-state consumption and savings for each type of agents, and for the steady-state value of L using the algorithm described in Section 4.1.4, and then one has to check the following ranking condition:

Condition

$$c_1 < c_{L0}$$

The previous condition is that highest steady-state consumption of unemployed agents if lower than the lowest steady-state consumption of employed agents. Indeed, the consumption of households just becoming unemployed (and thus being employed the previous period) c_1 is the higher consumption of unemployed agents, because consumption is falling with the length of the unemployment spell. Moreover, the consumption of employed agents who were at the credit constraint the previous period, c_{L0}, is the lowest consumption level of employed agents, because these agents have the lowest beginning-of-period wealth $-\bar{a}$. If the condition is fulfilled, one can always find a threshold c^* such that the period utility function is well behaved.

This framework can nevertheless be used in realistic dynamics models when applied to a subgroup of the population.

4.2.3 Using reduced-heterogeneity to model wealth inequality over the business-cycle

Challe and Ragot (2014) apply the previous framework to model the bottom 60% of the US households, based on the following observation. The wealth share of the poorest 60% of households in terms of liquid wealth is as low as 0.3%. Indeed, as the model is used to model precautionary saving in the business cycle, one should indeed focus on the net worth which can readily be used for short-run change in income. Define the period to be a quarter.

The modeling strategy is the following. Challe and Ragot (**?**) model the top 40% of the households by a family, which can fully insure its member against unemployment risk. This family as a discount factor $\beta^P$for patient and it has thus a standard Euler equation (without the employment risk, which is insured).

$$u'(c_t^P) = \beta^P \mathbb{E}(1 + r_{t+1}) u'(c_{t+1}^P)$$

As a consequence, the bottom 60% is modeled by agents having a quasi-linear utility function and having a lower discount factor, denoted as $\beta^I < \beta^P$ (I for impatient, P for patient). With such a low wealth shares of 0.3% (few hundred dollars of savings), it is easy to show that the households spend all their saving after a quarter of unemployment. This implies that one can construct an equilibrium for the bottom , where $L = 1$. In words, these households face the credit constraint after one period (one quarter) of unemployment.

The inter-temporal choice of employed agents can be simply written as

$$\eta \quad = \quad \beta^I \mathbb{E}(1 + r_{t+1}) \left(\alpha_{t+1} \eta + (1 - \alpha_{t+1}) u'(\delta + a_{0,t+1}(1 + r_{t+1}) + \bar{a}) \right),$$

where we used the fact that $c_{1,t} = \delta + a_{0,t}(1 + r_t) + \bar{a}$.

One can linearize the previous equation to obtain a simple saving rule. Level-deviation from the steady state are denoted with a hat. Linearization gives

$$\hat{a}_{0,t+1} = \Gamma_s \mathbb{E}(\hat{\alpha}_{t+1}) + \Gamma_r \mathbb{E}(\hat{r}_{t+1})$$

where Γ_s, Γ_r are coefficients which depend on parameter values. One can show that $\Gamma_s < 0$, because agents facing a higher probability to stay employed decrease their precautionary savings. The coefficient Γ_r can be either positive or negative depending on parameter values, depending on income and substitution effect.

Due to the saving rule, the model differs from hand-to-mouth DSGE models in the tradition of Kyotaki and Moore (1997). In the previous model, the number of credit constrained agents is very low, as households at the constraint are the fraction of impatient agent who are unemployed and not the full population of impatient agents. Finally, the conclusion of this model that poor households (the bottom 60% of the wealth distribution) react more to the unemployment risk than rich households is confirmed by households data (see Krueger at al. (2015)).

Finally, Challe and Ragot (2014) show that this model does a relatively good job in reproducing time-varying precautionary saving (compared to Krusell and Smith (1998)), and that the model is not more complicated than a standard DSGE model. In particular it, can be solved easily using DYNARE.

4.2.4 Other references and remarks

Le Grand and Ragot (2016c) extend this environment to introduce various segment in the period utility function to consider various type of agents. They apply this framework to show that such a model can reproduce a rich set of empirical moments when limited participation in financial market is introduced. In particular, the model can reproduce the low risk free rate, the equity premium the volatility of the consumption growth rate of the top 50% together with aggregate volatility of consumption.

The use of quasi-linear utility function provides (with the relevant set of assumptions) a simple representation of households heterogeneity focusing on the poor households who indeed face a higher unemployment risk. The cost of this representation is that the existence conditions can be violated for big aggregate shocks or for alternative calibration of the share of households facing the unemployment risk. As a consequence, this representation is not well-suited for Bayesian estimation, as it is not sure that existence conditions are fulfilled for any samples. To overcome this difficulty, other assumptions can be introduced.

4.3 Models based on a "family" assumption

The modeling strategy of previous models is based on the reduction of the state-space by reducing heterogeneity among a high-income agents and to set the credit constraints at a level higher than the natural borrowing limit to insure that low-income households reach the credit limit in a finite number of periods. The two previous modeling strategies played with utility functions to reach this goal. The last modeling strategy follows another route and consider directly different market arrangements for any utility function. It will be assumed that there is risk-sharing among employed agents. The presentation follows Challe, Matheron, Ragot, Rubio-Ramirez (2016), but it is much simpler because we do not introduce habit formation.

4.3.1 Assumptions

The model is based on limited insurance ("or the family assumption") often used in macro, but applied to a subgroup of agents. Assume that all agents belong to a family and that the family head care for all agents, but it has limited ability to transfer resources across agents. Indeed, agents are on islands and the planner can transfer resources across agents on the same islands, but it can not transfer resources across islands. More specifically:

1) All employed agents are on the same islands, where there are full risk sharing.

2) All unemployed agent for k periods leave on the sale island where there is full risk sharing.

3) A representative of the family head implement the consumption-saving choice in all islands, maximizing total welfare

4) The representative of the family head choose allocate wealth to households before knowing their next period employment status.

The structure can be seen as a deviation from Lucas (1990) to reduce risk-sharing among employed agents. It will generate the same structure as the previous models : no heterogeneity among employed agents and heterogeneous unemployed agents according to the length of their unemployment spell.

4.3.2 Equilibrium structure

As before, we use a guess-and-verify strategy to present the model. Assume that the unemployed agents reach the credit constraint after L consecutive periods of unemployment.

As before, denote as $n_{0,t}$ the number of employed agent at date t, and as $n_{k,t}$ the number of agents unemployed for $k \geq 1$ consecutive periods at date t. Denote as V^f as he value function of the family head, and as V^k the value function of households unemployed for k periods. These value functions can be written as

$$V^f\left(A_t^f, n_{0,t}, X_t\right) = \max_{S_{t+1}^f, c_{0,t}, l_t} n_{0,t} U(c_{0,t}, l_t) + \beta \mathbb{E}\left[V^f\left(A_{t+1}^f, n_{0,t+1}, X_{t+1}\right) + V^1\left(a_{0,t+1}, X_{t+1}\right)\right]$$

$$c_{0,t} + a_{0,t+1} = w_t l_t + \frac{A_t^f}{n_{0,t}}(1+r_t)$$

$$A_{t+1}^f = \alpha_{t+1} n_{0,t} a_{0,t+1} + (1-\rho_{t+1}) \sum_{k=1}^{L-1} n_{k,t} a_{k,t+1} - (1-\rho_{t+1}) n_t^c \bar{a}$$

Let's explain this problem. The family head maximizes the utility of all agents in the island, (imposing the same consumption and labor choice to all agents to maximize welfare). It takes into consideration the fact that some employed agents will fall into unemployment next period. The budget constraint in the previous problem is written in per capita term: Per capita income is equal to per capita consumption and per capita savings, denoted as $\frac{S^f_{t+1}}{n_{0,t}}$, where S^f_{t+1} is the total saving of the employed agents. At the end of the period, all agents in the "employed" agents have the same wealth, which is $\frac{S^f_{t+1}}{n_{0,t}}$'. As the family head cannot discriminate between agents before they leave the island, this will the next-period beginning-of-period wealth of agents leaving the island, $a_{1,t+1} = \frac{S^f_{t+1}}{n_{0,t}}$, i.e.. just falling into unemployment.

Finally, the next period wealth of employed agents is the next period pooling of the wealth of agents staying or becoming employed. First, it sums the wealth of agents staying employed $\alpha_{t+1}S^f_{t+1}$, and the wealth of unemployed agents not at the credit constraint in period t and finding a job in period $t+1$, $(1-\rho_{t+1})\sum_{k=1}^{L} n_{k,t}a_{k,t+1}$, and the wealth of agents constrained in period t and finding a job in period $t+1$. These agents have a wealth $-\bar{a}$ (Recall that n^c_t is the number of agents at the credit constraint in period t).

The value function of unemployed agents for $k$periods is simpler

$$
\begin{aligned}
V^k\left(a_{k-1,t}, X_t\right) &= \max_{a_{k,t+1},c_{k,t}} n_{k,t}U(c_{k,t},\delta) + \beta\mathbb{E}\left[V^f\left(A^f_{t+1}, n_{0,t+1}, X_{t+1}\right) + V^{k+1}\left(a_{k,t+1}, X_{t+1}\right)\right] \\
c_{k,t} + a_{k,t+1} &= \delta + a_{k-1,t}(1+r_t) \\
A^f_{t+1} &= \alpha_{t+1}S^f_{t+1} + (1-\rho_{t+1})\sum_{k=1}^{L-1} n_{k,t}a_{k,t+1} - (1-\rho_{t+1})n^c_t\bar{a} \\
a_{k,t+1} &\geq -\bar{a}
\end{aligned}
$$

In this maximization, the representative of the family head in the island where agents are unemployed for k periods take into account the fact that they will affect the next period wealth of employed agents, because all agents belong to a whole family, and the wealth of agents unemployed for $k+1$ periods, next period. First order and envelop conditions for employed agents are

$$\begin{aligned}
w_t U_1(c_{0,t}, l_t) &= -U_2(c_{0,t}, l_t) \\
U_1(c_{0,t}, l_t) &= \beta \mathbb{E}\left[\alpha_{t+1} V_1^f \left(A_{t+1}^f, n_{0,t+1}, X_{t+1}\right) + \frac{1}{n_{0,t}} V_1^1 \left(a_{0,t+1}, X_{t+1}\right)\right] \\
V_1^f \left(A_t^f, n_{0,t}, X_t\right) &= (1+r_t) U_1(c_{0,t}, l_t)
\end{aligned}$$

First order and envelop conditions for unemployed agents are

$$\begin{aligned}
n_{k,t} U_1(c_{k,t}, \delta) &= \beta \mathbb{E}\left[(1-\rho_{t+1}) n_{k,t} V_1 \left(A_{t+1}^f, n_{0,t+1}, X_{t+1}\right) + V_1^{k+1} \left(a_{k,t+1}, X_{t+1}\right)\right] \\
V_1^k \left(a_{k-1,t}, X_t\right) &= (1+r_t) n_{k,t} U_1(c_{k,t}, \delta)
\end{aligned}$$

Combining these equations give (and using the fact that $n_{1,t+1} = (1-\alpha_{t+1}) n_{0,t}$, and $n_{k+1,t+1} = \rho_{t+1} n_{k,t}$), one finds the set of equations defining the agents' choice.

$$\begin{aligned}
U_1(c_{0,t}, l_t) &= \beta \mathbb{E}(1+r_{t+1}) \left[\alpha_{t+1} U_1(c_{0,t+1}, l_{t+1}) + (1-\alpha_{t+1}) U_1(c_{1,t+1}, \delta)\right] && (14) \\
U_1(c_{k,t}, \delta) &= \beta \mathbb{E}(1+r_{t+1}) \left[(1-\rho_{t+1}) U_1(c_{0,t+1}, l_{t+1}) + \rho_{t+1} U_1(c_{k+1,t+1}, \delta)\right], \text{ for } k = 1, ..., L-1 \\
w_t U_1(c_{0,t}, l_t) &= -U_2(c_{0,t}, l_t) && (15) \\
A_{t+1}^f &= \alpha_{t+1} n_{0,t} a_{0,t+1} + (1-\rho_{t+1}) \sum_{k=1}^{L-1} n_{k,t} a_{k,t+1} - (1-\rho_{t+1}) n_t^c \bar{a} \\
c_{0,t} + a_{0,t+1} &= w_t l_t + \frac{A_t^f}{n_{0,t}} (1+r_t) && (16) \\
c_{k,t} + a_{k,t+1} &= \delta + a_{k-1,t}(1+r_t), \text{ for } k = 1, ..., L
\end{aligned}$$

Given the prices r_t and w_t, this is system of $2L+3$ equations for the $2L+3$ variables $((c_{k,t})_{k=0..L}, (a_{k,t})_{k=0..L-1} A_t^f, l_t)$. The key result of this construction is that the Euler equations of employed and unemployed agents are the same as the one obtained in a model with uninsurable idiosyncratic risk. Indeed, using the law of large numbers (which is assumed to be valid in a continuum), the "island" metaphor transforms idiosyncratic probabilities into shares of agents switching between islands. The gain is that the state-space is finite and the amount of heterogeneity is finite, as there are only $L+1$ different wealth levels.

Note that the consumption of $c_{k,t}$ for $k \geq L+1$ is $c_{k,t} = \delta + \bar{a} r_t$ because $a_{k,t+1} = \bar{a}$ for $k \geq L$.

The capital stock and total labor supply are simply

$$\begin{aligned} K_t &= \sum_{k=0}^{L-1} n_{k,t} a_{k,t} - n_t^c \bar{a} \\ L_t &= n_{0,t} l_t \end{aligned}$$

Compared to the environments in Section 4.1 and in Section 4.2, the current equilibrium exhibits less heterogeneity, as all employed agents consume and work the same amount. The gain is that the period utility function can be very general.

4.3.3 Algorithm and simulations

The algorithm to find the steady-state is the following.

1. Guess a value for $L \geq 1$
2. Guess a value for r, From r deduce w using the FOCs of the firms
3. Guess a value for c_0, deduce the labor supply using (15).
   (a) Solve for the consumption of agents $c_{k=0...L}$ using the Euler equations of the agents, from c_1 to c_L
   (b) Solve for the saving of the agents from a_0 to a_L using the budget constraint of all agents, and the values $c_{k=0...L}$.
   (c) Solve for the share of agents n_k for $k = 0, ..., L-1$ and n^c.
   (d) Find the aggregate capital stock K and aggregate labor L.
4. Iterate on r, until the financial market clears, i.e. until $r = \lambda K^{\lambda-1} L^{1-\lambda} - \mu$
5. Iterate on L, until

$$\begin{aligned} a_{L-1} &> -\bar{a} \\ u'(c_L) &> \beta(1+r)\left((1-\bar{\rho})\, u'(c_0) + \rho u'(c_{L+1})\right) \end{aligned} \tag{17}$$

   where $c_{L+1} = \delta - r\bar{a}$.

The model is again a finite set of equations, which can be again simulated using DYNARE. The DYNARE solver could be used to double-check the values of the steady state.

4.3.4 Example of quantitative work

Challe, Matheron, Ragot, Rubio-Ramirez (2016) use this representation of heterogeneity to construct a full DSGE model with heterogeneous agents. Indeed, the authors assume that only the bottom 60% in the wealth distribution form precautionary saving, and that the top 40% can be modeled by a representative agent.

They then introduce many other features to build a quantitative model, such as 1) sticky prices, 2) habit formation (what complexify significantly the exposition of the equilibrium) 3) capital adjusment costs, 4) search-and-matching frictions on the labor market, and 5) stochastic growth.

The general model is then brought to the data using Bayesian estimation. The information used in the estimation procedure includes thus the information set used to estimate DSGE models. In addition, information about time-varying consumption inequalities across agents can be used in the estimation process. The model is used to assess the role of precautionary saving during the great recession in the US. The authors show that a third of the fall in aggregate consumption can be attributed to time-varying precautionary saving due to the increase in unemployment during this period.

4.4 Assessment of small-heterogeneity models

The three classes of small-heterogeneity model presented in this Section have the merit to keep the effects of time-varying precautionary savings using perturbation methods. Compared no-trade equilibria, the actual quantity of assets used to self-insure, i.e. the optimal quantity of liquidity in the sense of Woodford (1990), is endogenous. In addition, they can easily be applied to a relevant subset of households in a DSGE model.

An additional gain of these representations is that the state space is small. For an equilibrium where agents hit the borrowing constraint after L periods of unemployment, there are only $L+1$ different wealth levels.

There are nevertheless two main drawbacks. First, when L grows, the state space doesn't converge toward a Bewley economy, because there is no heterogeneity across employed households. As a consequence, one can not consider the Bewley economy as the limit of these equilibria when there are no aggregate shocks. As a consequence, one cannot use all the information about the cross-section of households inequality (for instance all employed agents are similar). Hence, the models capture only a part of time-varying precautionary saving. Admittedly, a key part as the unemployment risk is the biggest uninsurable risk faced by households (Carroll et al. 2003).

The second drawback is that the number of periods of consecutive unemployment before the credit constraint binds (denoted as L in this Section) is part of the equilibrium definition : it has to be computed as a function of the model parameters. If aggregate shocks are small enough, this number is not time-varying, but this has to be checked during the simulations.

New developments provide environments without these drawbacks, at the cost of a bigger state space.

5 Truncated-history models

Le Grand and Ragot (2016b) present a general model to generate limited heterogeneity with an arbitrarily large but finite state space and which can be made close to the Bewley model. Le Grand and Ragot (2016b) construct an environment where the heterogeneity across agents depends only on a finite but possibly arbitrarily large number, denoted N, of consecutive past realizations of the idiosyncratic risk (as a theoretical outcome). As a consequence, the history of idiosyncratic risk is truncated after N periods. In this model, agents sharing the same idiosyncratic risk realizations for the previous N periods choose the same consumption and wealth. As a consequence, instead of having a continuous distribution of heterogeneous agents in each period, the economy is characterized by a finite number of heterogeneous consumption and wealth levels. The model can be simulated with DYNARE and optimal policy can be derived solving a Ramsey problem in this environment. The presentation follows the exposition of the decentralized equilibrium of Le Grand and Ragot (2016b)[4].

5.1 Assumptions

Truncated histories. For any N, each agent enters any period with the $N-$period history $\tilde{e}^N \in \mathcal{E}^N$, $\tilde{e}^N = \{\tilde{e}_{N-1}, ..., \tilde{e_0}\}$. This $N-$periods history is a truncation of the whole history of each agent. It is the history of any agents for the last N periods, before the agent learns its

[4]In the paper, Le Grand Ragot shows that this allocation can be represented as the allocation of a constrained planner. This insures existence and uniqueness for a given price dynamics.

current idiosyncratic shock $e \in \mathcal{E}$ for the current period. It then has the $N+1-$period history $e^{N+1} = (\tilde{e}^N, e)$ that we can also write $e^{N+1} = (e_N, e^N)$. In the former notation, e^{N+1} is seen as the the history $\tilde{e}^N$ with the successor state e, while in the latter notation, e^{N+1} is seen as the state e_N followed by the $N-$period history e^N).

The probability $\Pi_{\hat{e}^N,e^N}(X)$ that an household with history $\hat{e}^N = (\hat{e}_{N-1}, \ldots, \hat{e}_0)$ in the current period, experiences history $e^N = (e_{N-1}, \ldots, e_0)$ in the next period is the probability to switch from e state $\hat{e}_0$ in the current period to state e_0 in the next period, provided that histories $\hat{e}^N$ and e^N are compatible. More formally:

$$\Pi_{t,\hat{e}^N,e^N}(X) = 1_{e^N \succeq \hat{e}^N} M_{\hat{e}_0,e_0}(X), \tag{18}$$

where $1_{e^N \succeq \hat{e}^N} = 1$ if e^N is a possible continuation of history $\hat{e}^N$.

From the expression (18) of the probability $\Pi_{\hat{e}^N,e^N}$, we can deduce the dynamics of the number of agents having history e^N in each period, denoted S_{e^N} :

$$S'_{e^N} = \sum_{\hat{e}^N \in \mathcal{E}^N} S_{\hat{e}^N} \Pi_{\hat{e}^N,e^N}(X), \tag{19}$$

The previous expression is the application of the law of large number in a continuum.

Preferences. For quantitative reason that will appear clear below, we assume that the utility derived from consumption and leisure can depend on the history of idiosyncratic risk e^N, where recall that $e^{N+1} = (e_N, e^N)$. As a consequence, preference depends on the current and the last $N-1$ idiosyncratic shocks. The period utility is thus $\xi_{e^N} U(c,l)$, where $\xi_{e^N} > 0$.

To simplify the algebra, it is assumed that the period utility function exhibits no-wealth effect on the labor supply (what is consistent with empirical estimates). The period utility function is of the Greenwood-Hercowitz-Huffman (GHH) type

$$U(c,l) = u\left(c - \frac{l^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right)$$

State Vector. Denote X the state vector in each period, which is necessary to for rational expectations. This state vector will be specified below. For now it is sufficient to assume that it is finite dimensional.

Transfer. The trick to reduce heterogeneity is to assume that each agent receives a lump-sum transfer which depends on her $N+1-$history. This lump-sum transfer is denoted $\Gamma_{N+1}(e^{N+1}, X)$ and it will be balanced in each period.

Program of the agents. The agent maximizes her inter-temporal welfare by choosing the current consumption c, labor effort l and asset holding a'. She will have to pay an after-tax interest rate and wage rate denoted as r and w, as before. The value function can be written as

$$V(a, e^{N+1}, X) = \max_{a',c,l} \xi_{e^N} u\left(c - \frac{l^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right) + \beta \mathbb{E}\left[\sum_{e' \in \mathcal{E}} M_{e,e'}(X) V(a', (e^N, e'), X')\right] \tag{20}$$

$$a' + c = w(X) n_e l + \delta 1_{e=0} + (1 + r(X))a + \Gamma_{N+1}(e^{N+1}, X) \tag{21}$$

$$c, l \geq 0, a' \geq -\bar{a} \tag{22}$$

Denote $\tilde{\eta}(a, e^{N+1}, X)$ the Lagrange multiplier of the credit constraint $a' \geq -\bar{a}$. The solution to the maximization program (20)–(22) are the policy rules denoted $c = g_c(a, e^{N+1}, X)$, $a' = g_{a'}(a, e^{N+1}, X)$, $l = g_l(a, e^{N+1}, X)$ and the multiplier $\tilde{\eta}$ satisfying the following first order conditions, written in a compact form (I omit the dependence in X to lighten notations):

$$\xi_{e^N} u'\left(c - \frac{l^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right) U_c(c, l) + \tilde{\eta} = \beta \mathbb{E}\left[\sum_{e' \in \mathcal{E}} M_{e,e'} \xi_{e'^N} \left(c' - \frac{l'^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right)(1+r')\right], \tag{23}$$

$$l = (w n_e)^{\varphi}, \text{ if } e > 0, \tag{24}$$

$$l = \delta \text{ if } e = 0, \tag{25}$$

$$\tilde{\eta}(a' + \bar{a}) = 0 \text{ and } \tilde{\eta} \geq 0. \tag{26}$$

5.2 Equilibrium structure

We can show that all agents with the same current history e^N have the same consumption, saving and labor choices. To do so, we follow a guess-and-verify strategy. Assume that agents entering the period with a history $\tilde{e}^N$ before the idiosyncratic shock, have the save beginning-of-period saving $a_{\tilde{e}^N}$. These agents have a current productivity shock e, and have thus an history $e^{N+1} = (\tilde{e}^N, e) = (e_N, e^N)$. There are $S_{\tilde{e}^N,-1}$ agents with a beginning-of-period history $\tilde{e}^N$ and S_{e^N} agents with a current (i.e. after the current shock) history e^N.

Under the assumption that for any $\tilde{e}^N \in \mathcal{E}^N$, agents having the history $\tilde{e}^N$ have the same beginning-of-period wealth $a_{\tilde{e}^N}$, the average welfare (before transfer) of agents having a current N-period history is

$$\tilde{a}_{e^N} = \sum_{\tilde{e}^N \in \mathcal{E}^N} \frac{S_{\tilde{e}^N}}{S'_{e^N}} \Pi_{\tilde{e}^N, e^N} a_{\tilde{e}^N}, \text{ for all } e^N. \tag{27}$$

The term $\sum_{\tilde{e}^N \in \mathcal{E}^N} S_{\tilde{e}^N} \Pi_{\tilde{e}^N, e^N} a_{\tilde{e}^N}$ is the total wealth of agents having current history e^N. Dividing by the number of those agents, we find the per capita value $\tilde{a}_{e^N}$. The transfer is now easy to define:

$$\Gamma_{N+1}(e^{N+1}, X) \equiv (1+r)\left(\tilde{a}_{e^N} - a_{\tilde{e}^N}\right). \tag{28}$$

The transfer Γ_{N+1} swaps the remuneration of the beginning-of-period wealth $a_{\tilde{e}^N}$ of agents having history $\tilde{e}^N$ by the remuneration of the average wealth $\tilde{a}_{e^N}$ of agents having the current $N-$period history $e^N = (\tilde{e}_{N-2}, \ldots, \tilde{e}_0, e)$. As a consequence, this transfer is balanced.

The impact of transfer on agents' wealth. It is easy to see that all agents consider the lump-sum transfer Γ_{N+1} as given and thus do not internalize the effect of their choice on this transfer (because there is a continuum of agents for any truncated history). We consider the impact of transfer $\Gamma_{N+1}(e^{N+1}, X)$ for an agent with history $e^{N+1} = (\tilde{e}^N, e) = (e_N, e^N) \in \mathcal{E}^{N+1}$ –we see e^{N+1} as having either the predecessor $\tilde{e}^N$ or the successor e^N— and beginning-of-period wealth $a_{\tilde{e}^N}$. Her budget constraint (21) can be expressed using transfer expression (28) as follows:

$$a' + c = wn_e l + \delta 1_{e=0} + (1+r)\tilde{a}_{e^N}. \tag{29}$$

The beginning-of-period and after-transfer wealth of agents with history e^{N+1} only depends on the current $N-$period history e^N. Moreover, as can be seen from (20), agents with the same $N-$period history e^N are endowed with the same expected continuation utility as long as they save the same amount a'. Therefore, agents with the same current $N-$period history e^N behave similarly: they consume the same level, they supply the same labor quantity, and hold the same wealth.

State vector. The aggregate state of the economy X is the collection of: (i) the beginning-of-period wealth distribution depending on the $N-$period history $(S_{e^N}, a_{e^N})_{e^N \in \mathcal{E}^N}$ (i.e., the size of agent population with history e^N, together with their respective wealth), and (ii) the aggregate state h, which affects transition probabilities.

5.3 Equations of the model

We can now write the equations of the models with a truncation for N periods of idiosyncratic histories. We present it in sequential form to simplify the reading and the numerical implementation. Define as $\mathcal{C}_t$, the set of $N-$period idiosyncratic histories for which agents face credit constraints (We show how to find $\mathcal{C}_t$ below). The histories $\mathcal{E}^N - \mathcal{C}_t$ are not credit constrained.

$$a_{t,e^N} + c_{t,e^N} = w_t l_{t,e^N} + \delta 1_{e_t^N=0} + (1+r_t)\tilde{a}_{t,e^N}, \text{ for all } e^N, \tag{30}$$

$$\xi_{e^N} U_c\left(c_{t,e^N}, l_{t,e^N}\right) = \beta \mathbb{E}_t \left[\sum_{\hat{e}^N \in \mathcal{E}^N} \Pi_{t,e^N,\hat{e}^N} \xi_{\hat{e}^N} U_c\left(c_{t+1,\hat{e}^N}, l_{t+1,\hat{e}^N}\right)(1+r_{t+1})\right], \text{ for } e^N \in \mathcal{E}^N - \mathcal{C}_t \tag{31}$$

$$l_{t,e^N} = w_t^{\varphi} 1_{e_t>0} + \delta 1_{e_t=0}, \text{ for all } e^N, \tag{32}$$

$$a_{t,e^N} = -\overline{a}, \text{ for } e^N \in \mathcal{C}_t \tag{33}$$

$$\tilde{a}_{t,e^N} = \sum_{\tilde{e}^N \in \mathcal{E}^N} \frac{S_{t-1,\tilde{e}^N}}{S_{t,e^N}} \Pi_{t-1,\tilde{e}^N,e^N} a_{t-1,\tilde{e}^N}, \text{ for all } e^N, \tag{34}$$

$$S_{t,e^N} = \sum_{\tilde{e}^N \in \mathcal{E}^N} S_{t-1,\tilde{e}^N} \Pi_{t-1,\tilde{e}^N,e^N}, \text{ for all } e^N, \tag{35}$$

$$K_{t+1} = \sum_{e \in \mathcal{E}^N} S_{t,e^N} a_{t,e^N}, \tag{36}$$

$$L_t = \sum_{e \in \mathcal{E}^N} S_{t,e^N} l_{t,e^N}, \tag{37}$$

$$r_t = \lambda A_t K_t^{\lambda-1} L_t^{1-\lambda} - \mu \tag{38}$$

$$w_t = (1-\lambda) A_t K_t^{\lambda} L_t^{-\lambda} \tag{39}$$

For a given technology process A_t, this system has $5 \times E^N + 4$ equations for $5 \times E^N + 4$ variables $((a_{t,e^N}, c_{t,e^N}, l_{t,e^N}, \tilde{a}_{t,e^N}, S_{t,e^N})_{e^N \in \mathcal{E}^N}, K_{t+1}, r_{t+1}, w_t, L_t)_{t=0\ldots\infty}$. Before simulating the model, one has to find the set of histories facing credit constraint $\mathcal{C}_t$. To so so, we first solve the model under steady state to find $\mathcal{C}$, and then we assume that shocks are small enough to check that credit constraints are indeed binding.

5.4 Algorithm for the Steady state

For a given N, rank first the agents according to their idiosyncratic history. For instance, for the history $e^{N,t} = \{e_{N-1}, ..., e_0\} \in \mathcal{E}^N$, one can consider $k = 1+\sum_{i=0}^{N-1} e_i E^{N-1-i}$. With this ordering, $k = 1$ for the agents who are unemployed for N periods $e^t = \{0, ..., 0\}$ and $k = E^N = 2^N$ for $e^t = \{1, ..., 1\}$. In words, agents with a low k have been unemployed recently for a long period of time. All the steady-state variables will be indexed by k instead of e^N. We thus look for $((a_k, c_k, l_k, \tilde{a}_k, S_k)_{k=1..E^N}, K, r, w, L)$.

The algorithm for the steady-state is the following.

1. Assume that the set of histories S^c is credit constrained. For all $k \in S^c$, we have $a_k = -\overline{a}$.
   (a) Consider an interest r such that $\beta(1+r) < 1$. Deduce the real wage w, using equations (38) and (39).
   (b) Deduce l_k using (32).
      i. Assume values for the vector of consumption levels of constrained agents $c_k, k \in S^c$.
      ii. Using the Euler equations (31) and labor supply l_k find the consumption levels of unconstrained agents $c_k, k \notin S^c$.
      iii. Using the budget constraints (30) of unconstrained agents $k \notin S^c$ and the risk-sharing equation (34) find the savings of unconstrained agents. $a_k, k \notin S^c$
      iv. Deduce from the budget constraint of constrained agents, $k \in S^c$, the implied consumption levels $\tilde{c}_k, k \in S^c$
      v. Iterate over $c_k, k \in S^c$ until $c_k = \tilde{c}_k, k \in S^c$.
   (c) compute the implied interest rate $\tilde{r}$ using equations (36), (37) and (38).
   (d) Iterate on $r$until $r = \tilde{r}$
2. Check that histories $k \in S^c$ are credit constrained and histories $k \notin S^c$ are not, iterate on S^c otherwise.

5.5 Dynamics

The model can be easily simulated for small aggregate shocks at the first order, assuming that those shocks are small enough such that constrained histories remain always the same. This can be check during each simulation, checking that Euler equations hold with inequality for assumed constrained households. The simples way to simulate the model is 1) to write a code which writes the set of equations (30) - (39) as a DYNARE code 2) Specify the steady-state found in the previous Section as initial values to double check that the steady-state is correct, using the “resid” function of DYNARE 3) Simulate the model using “stoch_simul” function.

5.6 Choosing the preference shifters ξ_{e^N}

The simulations of the model can be done for any ξ_{e^N}. How can we choose these parameters ? What are they useful for? The simplest choice is to set $\xi_{e^N} = 1, e^N \in \mathcal{E}^N$. But the choice of these ξ_{e^N} can be made to improve the fit of the equilibrium distribution to a given target. Indeed, if one has empirical average estimates of wealth levels of agents for observed history on the labor market e^N, namely $\hat{a}_{e^N}$ for $e^N \in \mathcal{E}^N$, then one can iterate over ξ_{e^N}, until the steady savings values of the model are close enough to their empirical counterpart $a_{e^N} \simeq \hat{a}_{e^N}$ for $e^N \in \mathcal{E}^N$. For instance, one can simulate a Bewley-Aiyagari-Hugget model to get the model generated averages $\hat{a}_{e^N}$ for $e^N \in \mathcal{E}^N$, and then iterate over ξ_{e^N} in the truncated economy for the steady-state outcome of the truncated model to be close to the “true” values $\hat{a}_{e^N}$ for $e^N \in \mathcal{E}^N$.

5.7 Numerical example

As an example, one can now provide numerical simulations of the truncated model. The goal of this simulation is not to provide a quantitatively relevant model, but to show how the truncated economy can be simulated. First, assume that the period GHH - utility function is

$$U(c,l) = \frac{1}{\sigma-1}\left(c - \frac{l^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right)^{\sigma-1},$$

where φ is the inter-temporal Frisch elasticity of labor supply. The discount factor is set equal to $\beta = 0.96$. The curvature of the utility function is $\sigma = 2$. The Frisch elasticity of labor supply is also set to $\varphi = 0.1$. The quantity λ is the capital share, which is set to 0.36, while μ is the annual depreciation rate, set to 10%. The replacement ratio is set to $\delta/w = 0.3$, which is in the lower range of empirical estimate. The credit constraint is set to $\bar{a} = 0$. As a benchmark the model is solved with $\xi_{e^N} = 1, e^N \in \mathcal{E}^N$.

N	β	φ	σ	μ	λ	ρ^a	σ^a	$\bar{a}$
6	0.96	1	2	0.1	0.36	0.8	0.01	0

Table 1: Parameter Values

	K	L	GDP	C^{tot}	r	w
Mean	4.07	0.95	1.60	1.21	.0417	1.08
St. dev	0.12	0.002	0.04	0.02	.003	0.025

Table 2: First and second order moments of key variables

The autocorrelation of the technology shock ρ^a is 0.8, and the standard deviation $\sigma^a = 0.01$. Table 1 summarizes parameter values.

Concerning the the labor process, the following transition matrix is considered

$$M = \begin{bmatrix} 0.2 & 0.8 \\ 0.05 & 0.95 \end{bmatrix}$$

This implies that the steady-state unemployment rate is roughly 6%.

The model is solved for $N = 6$, this implies that agents differ according to their idiosyncratic histories for the last 6periods and that there are $2^6 = 64$ different agents in this economy. As a software as DYNARE can handle few thousand equations, it implies that a maximum length of $N = 12$ (such that $2^{12} = 4096$) seems a maximum number with current standard computers.

In the steady state without aggregate shock, one finds that only agents unemployed for $N = 6$ periods are credit constrained. All other agents have a positive saving. The simulation of the model with aggregate shocks take 3 seconds in DYNARE.

The next table provides first and second order moments for this economy. The gain of using perturbation method and DYNARE is that impulse response function can be easily simulated, as the state space for the aggregate variable is continuous. As an example the following figure plots IRFs after a TFP shock (first panel), for relevant variables. The variable are provided in percentage proportional deviation from steady state values, except the real interest rate which is in percentage level deviation from its steady state value.

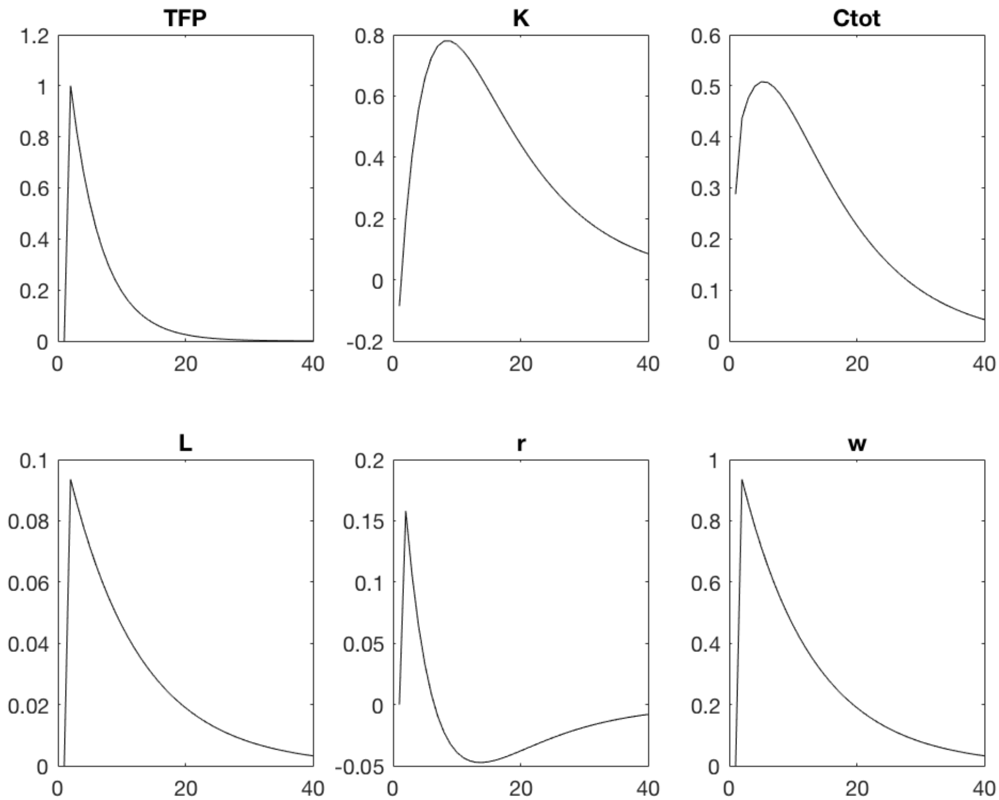

Figure 1: Aggregate IRF after a 1% increase in TFP

6 Optimal policies

A common outcome in the previous models is that one has a finite number of types of agents to follow, as an equilibrium outcome. This allows deriving optimal policies using Ramsey techniques developed in the representative agents environments (See Sargent and Ljungqvist 2012 for a textbook presentation). Ragot (2016) investigate optimal monetary policy in a model with capital accumulation and flexible prices. Bilbiie and Ragot (2017) study optimal monetary policy in a monetary model with sticky-prices. Challe (2017) analyze optimal monetary policy in a no-trade model with unemployment risk.

Recently Le Grand and Ragot (2016b) develop a general techniques to solve for optimal policies in truncated economies. They study optimal fiscal policies in a model with heterogeneous agents and aggregate shocks, with four instruments: a positive transfer, two distorting taxes on capital and labor and public debt. This is a promising route, as the distortions generated by incomplete insurance markets are hard to identify (See Aiyagari (1995) and Davila, Hong, Krusell and Rios-Rull (2012) for a contribution about the optimality of the level of the capital stock).

7 Comparison with other approach using perturbation methods

It maybe useful to compare the methods presented in this chapter with other methods using perturbation techniques. The key similarity of methods of this chapter is to use perturbation methods around a steady state with idiosyncratic risk but with a finite state-space support, as an equilibrium outcome. In addition, the state space for the aggregate risk has a infinite dimension (i.e. the aggregate state belongs to a continuum) as in the DSGE literature. The model can be then simulated with a finite number of equations using perturbations, such that software as DYNARE can be used.

Other strategies have been used to use perturbation methods[5]. First, Reiter (2009) uses perturbation methods to solve for aggregate dynamics around a steady-state equilibrium of a Bewley model, with idiosyncratic shock but no aggregate shock. These techniques, used for instance in McKay and Reis (2016), linearize policy rule around an equilibrium distribution which has an infinite support.

Mertens and Judd (2012) use perturbation method around a steady-state with neither aggregate nor idiosyncratic shock. They use a penalty function to pin down steady-state portfolios (which are not determined without risks). Other papers using perturbation method but restricting exogenously the state-space are Preston and Roca (2007) and Kim, Kollmann, and Kim (2010). For instance, one can solve the model using an exogenous finite grid for the saving decisions.

The techniques presented in this chapter implied that the reduction in heterogeneity appears has an equilibrium outcome. To summarize the previous remarks, the gain is threefold. First, this allows to consider many assets. For instance, Challe, Le Grand and Ragot (2013) price

[5] Global methods are also used to solve these models. See Algan, Allais, DenHaan, Rendahl (2014) for a survey of methods and Krueger, Dirk Mitman, Kurt and Perri, Fabrizio (**?**) for a recent application to the subprime crisis in the US. Den Haan (2014) discusses (global) solution methods for models with and without rational expectations.

arbitrarily large number of maturities of the yield curve. Le Grand and Ragot (2016a) price both assets and derivative assets in the same model.

Second, having a finite number of equations (in a possibly large model) allows using econometric techniques, such as Bayesian estimation, which are then relatively easy to implement. This is done in Challe, Matheron, Ragot and Rubio-Ramirez (2016) to quantify the contribution of precautionary saving to the business cycle in the US. Third, the finite number of equations allow deriving optimal policies in these environments.

The cost of using the models presented in this Chapter is that the representation of inequality is simplifies, in any case, compared to full-fledged Bewley models. As a consequence, these models are less useful when one wants to describe in details inequalities generated by self-insurance for uninsurable risks.

8 Heterogeneous expectations

The previous models have been presented under the assumption that agents share the same information set and form rational expectations. As a consequence, they all have the same expectations about aggregate variables. This outcome is rejected by data surveys, which shows heterogeneity in expectation about aggregate variables (see Carroll (2003), Branch (2004), and Massaro (2013)) heterogeneity in expectation formation for the same information set (See Hommes (2011) and Coibion and Gorodnichenko (2015)).

There is a vast literature on heterogeneous agent models in economics and finance where agents differ according to their expectations and their expectation formation. Hommes (2006) provide a survey of this literature. One can thus observe two literatures evolving in parallel analyzing heterogeneous agents. The first one, which is the subject of the current paper is about heterogeneity coming from uninsurable income risk under rational expectations. The second one is about behavioral models with boundedly rational agents, surveyed in Hommes (2006).

The tools developed in this chapter could be useful to model heterogeneous expectations in a tractable way. First, in some models agents choose in each periods the rules to form expectations (Branch and McGough (2009) and Massaro (2013) or how much effort to invest to form expectation (as in the rational inattention literature). As the various equilibrium structures presented in the previous Sections don't depend on rational expectations to reduce heterogeneity, they could be useful to construct models with a finite number of different expectations. Models where expectations are a state variable, as in learning models, may also be consistent with previous models. For instance, if agents experiencing the same history of shocks for a long period of time converge to the same expectations, then a truncation in histories of idiosyncratic shocks could also be a satisfactory assumption. Finally, as noted by Den Haan (2014) the case where some agents form rational expectations whereas others are boundedly rational are difficult to solve. The models presented here could allow to reduce heterogeneity among rational agents and could help to solve those models. The goal of these remarks is to indicate that using the tools presented in this Chapter to contribute to the literature on heterogeneous expectations seems a promising route.

9 Concluding remarks

This chapter surveys the heterogeneous agents literature, based on uninsurable idiosyncratic shocks. Agents differ according to "shocks" defined in a broad sense that occur in their life. We focused more precisely on representations of reduced heterogeneity with rational expectations which can be solved with perturbation methods. This allows introducing many other "frictions" studied in the literature, such as search-and-matching in the labor market, sticky prices, heterogeneity in skills, habit formation and son on. These models can be estimated, using a vast information set, including times-series and cross-sectional information. Although these models can match the data in many dimensions, they are less empirically relevant to account for expectations heterogeneity, which is key in alternative models of such as Agent-based models. These two literatures follow different empirical strategy such that it is difficult to compare them on empirical ground. In any case, integrated models with both uninsurable shocks, a role for forward-looking behavior together with relevant heterogenous expectations would be a good laboratory to quantify the relative importance of these various ingredients.

Finally, the models of this chapter has focused on households heterogeneity, leaving aside other forms of heterogeneity among firms or financial intermediaries. The tools developed in this chapter could easily be applied to these institutions. This opens a deeper question about the relevant level of aggregation to think about heterogeneity, and which type of heterogeneity matters for economic analysis. if a recent consensus seems to emerge about the importance of households heterogeneity, interactions of various types of heterogeinty in a dynamics set-up is a fascinating line of research.

References

Aiyagari, R. S. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, 109(3), 659–684.

Aiyagari, S. R. (1995): "Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints, and Constant Discounting," *Journal of Political Economy*, 103(6), 1158–1175.

Algan, Y., O. Allais, W. DenHaan, and P. Rendahl (2014): "Solving and Simulating Models with Heterogeneous Agents and Aggregate Uncertainty," in *Handbook of Computational Economics*, vol. 3, chap. 6, pp. 277–324. Elsevier.

Algan, Y., E. Challe, and X. Ragot (2011): "Incomplete Markets and the Output-Inflation Tradeoff," *Economic Theory*, 46(1), 55–84.

Bilbiie, F., and X. Ragot (2017): "Monetary Policy and Inequality when Aggregate Demand Depends on Liquidity," CEPR Working paper, Centre for Economic Policy Research.

Branch, W. (2004): "The theory of rationally heterogeneous expectations: evidence from survey data on inflation expectations," *The Economic Journal*, 114(497), 592?621.

Branch, W., and B. McGough (2009): "Monetary policy in a new Keynesian model with heterogeneous expectations," *Journal of Economic Dynamics and Control*, 33(8), 1036?1051.

Carroll, C. (2003): "Macroeconomic expectations of households and professional forecasters," *Quarterly Journal of Economics*, 118, 269?298.

Carroll, C. D., K. E. Dynan, and S. D. Krane (2003): "Unemployment Risk and Precautionary Wealth: Evidence from Households' Balance Sheets," *The Review of Economics and Statistics*, 85(3), 586–604.

Challe, E. (2017): "Uninsured unemployment risk and optimal monetary policy," Working paper.

Challe, E., F. LeGrand, and X. Ragot (2013): "Incomplete Markets, Liquidation Risk and the Term Structure of Interest Rates," *Journal of Economic Theory*, 148(6), 2483–2519.

Challe, E., J. Matheron, X. Ragot, and J. Rubio-Ramirez (2016): "Precautionary saving and Aggregate Demand," *Quantitative Economics, Forthcoming.*

Challe, E., and X. Ragot (2014): "Precautionary Saving Over the Business Cycle," *Economic Journal*, forthcoming.

Chetty, R., A. Guren, D. Manoli, and A. Weber (2011): "Are Micro and Macro Labor Supply Elasticities Consistent? A Review of Evidence on the Intensive and Extensive Margins," *American Economic Review*, pp. 1–6.

Chodorow-Reich, G., and L. Karabarbounis (2014): "The Cyclicality of the Opportunity Cost of Employment," Working paper, Harvard University.

Coibion, O., and Y. Gorodnichenko (2015): "Information Rigidity and the Expectations Formation Process: A Simple Framework and New Facts," *American Economic Review*, 105(8), 2644–2678.

Constantinides, G. M., and D. Duffie (1996): "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 104(2), 219–240.

Davila, J., J. Hong, P. Krusell, and J.-V. Rios-Rull (2012): "Constrained Efficiency in the Neoclassical Growth Model With Uninsurable Idiosyncratic Shocks," *Econometrica*, 80(6), 2431–2467.

Den Haan, W. (2014): "Solving Models with Heterogeneous Expectations," *Presentation slides.*

Fishburn, P. (1977): "Mean-Risk Analysis with Risk Associated with Below-Target Returns," *American Economic Review*, 67(2), 116–126.

Gornemann, N., K. Kuester, and M. Nakajima (2012): "Monetary Policy with Heterogeneous Agents," Working Papers 12-21, Federal Reserve Bank of Philadelphia.

Heathcote, J. (2005): "Fiscal Policy with Heterogeneous Agents and Incomplete Markets," *Review of Economic Studies*, 72(1), 161–188.

Heathcote, J., K. Storesletten, and G. Violante (2009): "Quantitative Macroeconomics with Heterogeneous Households," *Annual Review of Economis*, 1, 319–354.

Heathcote, J., K. Storesletten, and G. Violante (2014): "Optimal Tax Progressivity: An Analytical Framework," NBER Working Paper 19899, National Bureau of Economic Research.

Hommes, c. (2006): "Heterogeneous agent models in economics and finance," in *Handbook of Computational Economics*, vol. 2, pp. 1109–1186. Elsevier.

Hommes, C. (2011): "The heterogeneous expectations hypothesis: some evidence from the lab," *Journal of Economic Dynamics and Control*, 35, 1–24.

Huggett, M. (1993): "The Risk Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control*, 17(5-6), 953–969.

Kaplan, G., B. Moll, and G. Violante (2016): "The world according to HANK," *Mimeo.*

Kim, S. H., R. Kollmann, and J. Kim (2010): "Solving the Incomplete Market Model with Aggregate Uncertainty Using a Perturbation Method," *Journal of Economic Dynamics and Control*, 34(1), 50–58.

Kiyotaki, N., and J. Moore (1997): "Credit Cycles," *Journal of Political Economy*, 105(2), 211–48.

Krueger, D., K. Mitman, and F. Perri (2015): "Macroeconomics and Heterogeneity, Including Inequality," Mimeo, University of Pennsylvania.

Krusell, P., T. Mukoyama, and A. Sahin (2010): "Labour-Market Matching with Precautionary Savings and Aggregate Fluctuations," *Review of Economic Studies*, 77(4), 1477–1507.

Krusell, P., T. Mukoyama, and A. A. J. Smith (2011): "Asset Prices in a Huggett Economy," *Journal of Economic Theory*, 146(3), 812–844.

Krusell, P., and A. A. Smith (1998): "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 106(5), 867–896.

Lagos, R., and R. Wright (2005): "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113(3), 463–484.

LeGrand, F., and X. Ragot (2016a): " Incomplete Markets and Derivative Assets," *Economic Theory*, 62(3), 517–545.

——— (2016b): " Optimal policy with heterogeneous agents and aggregate shocks: An application to optimal public debt dynamics," *Working Paper*.

——— (2016c): "Asset returns, idiosyncratic and aggregate risk exposure," *Working Paper*.

Lucas, R. J. (1990): "Liquidity and Interest Rates," *Journal of Economic Theory*, 50(2), 237–264.

Massaro, D. (2013): "Heterogeneous expectations in monetary DSGE models," *Journal of Economic Dynamics and Control*, 37, 680–692.

McKay, A., and R. Reis (2016): "The Role of Automatic Stabilizers in the U.S. Business Cycle," Discussion Paper 84.

McKay, Alister., N. E., and J. Steinsson (2016): "The Discounted Euler Equation," *Mimeo Boston University and Columbia University*.

Mertens, T. M., and K. L. Judd (2012): "Equilibrium Existence and Approximation for Incomplete Market Models with Substantial Heterogeneity," NYU Working Paper 2451/31656, New York University.

Miao, J. (2006): "Competitive Equilibria of Economies with a Continuum of Consumers and Aggregate Shocks," *Journal of Economic Theory*, 128(1), 274–298.

Preston, B., and M. Roca (2007): "Incomplete Markets, Heterogeneity and Macroeconomic Dynamics," NBER Working Papers 13260, National Bureau of Economic Research.

Ragot, X. (2016): "Money and Capital Accumulation over the Business Cycle," Working Paper, SciencesPo.

Ravn, M., and V. Sterk (2013): "Job Uncertainty and Deep Recessions," Mimeo, University College London.

Reiter, M. (2009): "Solving Heterogeneous-Agent Models by Projection and Perturbation," *Journal of Economic Dynamics and Control*, 33(3), 649–665.

Rios-Rull, J.-V. (1995): "Models with heterogeneous agents," in *In Frontiers of Business Cycle Research*, ed. by C. TF., p. 16. Princeton, NJ: Princeton University Press.

——— (1997): "Computation of Equilibria in Heterogeneous Agent Models," *FRB of Minneapolis Research Department Staff Report 231*, (231).

Sargent, T. J., and L. Ljungqvist (2012): *Recursive Macroeconomic Theory*. MIT Press, third edn.

Scheinkman, J., and L. Weiss (1986): "Borrowing Constraints and Aggregate Economic Activity," *Econometrica*, 54(1), 23–45.

Tauchen, G. (1986): "Finite State Markov Chain Approximations to Univariate and VectorAutoregressions," *Economics Letters*, pp. 177–81.

Woodford, M. (1990): "Public Debt as Private Liquidity," *The American Economic Review*, 80(2), 382–388.