Interest rate and existence of stationary equilibria in incomplete insurance-market economies

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Abstract

We show that the difference between the interest rate and the discount rate is proportional to the severity of credit constraints at any stationary equilibrium of standard heterogeneous-agent economies. This severity is measured as the economy-wide average of the shadow price of the credit constraint. We deduce that stationary equilibria either feature binding credit constraints for a positive mass of agents or do not exist when credit constraints never bind. This has important implications for both positive and normative results in heterogeneous-agent models.

Keywords: Incomplete markets, interest rate, existence.

JEL codes: E21, E44, D91, D31.

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1 Introduction

Incomplete insurance market economies are becoming the workhorse setup for macroeconomic analysis. Such frameworks are also called Bewley-Huggett-Aiyagari-Imohoroğlu economies, named after the seminal papers of Bewley (1983), Huggett (1993), Imrohoroğlu (1992), and Aiyagari (1994) – or in short heterogeneous-agent economies. These economies have the great advantage of being able to reconcile a sound theoretical model with the actual heterogeneity observed in micro-data. However, these frameworks are very complex and not all theoretical aspects – among which, equilibrium existence and multiplicity – are fully understood yet. This article provides a novel characterization result stating that, at any stationary equilibrium, the difference between the discount rate $1/\beta$ and the interest rate $1+r$ is proportional to the average credit constraints' severity in the economy. This severity is measured by the average shadow price (i.e., Lagrange multiplier) of the households' borrowing constraint, where the average is computed over the whole population.

This characterization has several implications. First, the credit constraints' severity and the average marginal utility of households are two sufficient statistics to pin down the liquidity premium $\frac{1}{\beta} - (1+r)$. Second, if no positive mass of households experience binding credit constraints, the liquidity premium is null and the stationary equilibrium is characterized by $\beta(1+r) = 1$. As shown in Chamberlain and Wilson (2000), this implies that the stationary equilibrium does not exist in standard cases. Stationary equilibria in heterogeneous-agent models therefore go hand-in-hand with binding credit-constraints.

The setup of our proof consists of a standard heterogeneous-agent economy, where agents, facing a Markovian idiosyncratic productivity risk and borrowing constraints, can save in a riskless asset. The proof itself is rather straightforward and relies on the aggregation of all individuals Euler equations in a stationary equilibrium for general utility functions. The core of the paper focuses on a finite-state idiosyncratic risk to keep the presentation short. We also discuss several extensions, including a continuous space for the idiosyncratic risk, a more general income process, or capital taxes. All these extensions leave our results – including the non-existence one – unchanged. Overall, our characterization of the relationship between interest rate and credit constraints is very general and holds as long as households have access to an asset, whose return is not affected by their idiosyncratic risk.

Our result is reminiscent from Krebs (2004). He shows that no recursive equilibrium can exist in an incomplete-market economy – without production – with two agents if credit constraints do not bind. Several technical assumptions must also hold: the per period utility function must be unbounded from below (which rules out for instance CARA utilities and CRRA utilities with an elasticity of substitution above 1) and endowment process must follow a Markov chain with finite support. We consider a standard heterogeneous-agent economy (with a continuum of agents), with standard utility function (strictly increasing, concave) and production. Our result holds for any stationary equilibrium (independently of a recursive or a sequential formulation) and can easily be extended to a large class of income process. Furthermore, we also provide a robust relationship between interest rate and credit constraints in any stationary equilibrium economy.

Another related paper is Miao (2002), who proves the existence of a stationary recursive competitive equilibrium in a heterogeneous-agent economy with production. He also shows that whenever the equilibrium exists, the credit constraints must bind for a positive measure of agents. Miao's analysis however relies on some certain assumptions (smoothness condition on the Markov process, upper bound on the utility function, curvature assumption on the utility function). We relax all these assumptions and further show that any stationary equilibrium, if it exists, must feature binding credit constraints for a positive mass of agents, whenever individual income is stochastic (in the spirit of Chamberlain and Wilson 2000).

Finally, Açikgöz (2018) provides existence results of stationary equilibria when credit constraints bind in equilibrium. Our analysis show that this appears to be a complete charaterization of stationary equilibria in standard incomplete market economies.[1]

The rest of the paper is organized as follows. Section 2 presents the environment. Section 3 states our main characterization result and provides its proof in the case of a sequential formulation and finite-state idiosyncratic risks. Section 4 presents several extensions, while Section 5 discusses the implications for the literature.

2 Environment

The environment we consider is a standard heterogeneous-agent economy with production in discrete time. We assume that the economy is populated by a continuum of agents with a unit mass, and distributed on an interval $\mathcal{I}$ according to a measure $\ell(\cdot)$. We follow Green (1994) and assume that the law of large numbers holds.[2]

2.1 Idiosyncratic risk structure

In each period, every agent inelastically supplies one unit of labor and receives an income in exchange of her labor supply. This income, denoted by e_t, is risky and this risk cannot be insured nor avoided. Income realizations belong to a set denoted by E that is assumed

[1] Acemoglu and Jensen (2015) provide comparative statics results for standard incomplete-market economies. They prove equilibrium existence for an exogenously bounded set for asset choices. Our result is derived in the standard case that does not feature such bound on individual saving choices. It could be rather straightforward to extend the results to their setup, in which the equilibrium interest rate would depend on the mass of agents at the upper bound of the choice set.

[2] See also Miao (2006) for a careful treatment of the law of large numbers in these economies.

to be finite and to contain only distinct income possible realizations.[3] Furthermore, the income process follows a finite-state first-order Markov chain and we denote by $\Pi_{e,e'}$ the (constant) probability to switch from the current income $e \in E$ to the income $e' \in E$ in the next period.

The history of idiosyncratic shocks from date 0 to date t is denoted $e^t = \{e_0, \ldots, e_t\} \in E^{t+1}$ and gathers all shock realizations prior to date t. We denote by $e^{t+1} \succeq e^t$ the fact that the history e^{t+1} is a possible continuation of the history e^t, meaning that the realizations of e^t and e^{t+1} coincide for all dates from 0 to t. Finally, the quantity $\tilde{\Pi}_{e^t,e^{t+1}}$ represents the probability to switch from history e^t at t to history e^{t+1} at $t+1$. This probability is equal to the transition probability from e_t at t to e_{t+1} at $t+1$ if e^{t+1} is a continuation of e^t. Formally: $\tilde{\Pi}_{e^t,e^{t+1}} \equiv \Pi_{e_t,e_{t+1}} 1_{e^{t+1} \succeq e^t}$, where $1_{e^{t+1} \succeq e^t} = 1$ if $e^{t+1} \succeq e^t$ and 0 otherwise.

We now turn to the description of agents' distribution as a function of their idiosyncratic histories. To start with, note that we allow the initial distribution of agents' (at date 0) to depend on the agent's initial income. This assumption offers generality and enables the economy to possibly start from the steady-state wealth distribution. We denote by $A = [-\underline{a}, \infty)$ the set of possible wealth levels. Indeed, agents are prevented from borrowing more than $\underline{a} \geq 0$, which bounds from below the set A. Note that this borrowing limit – as any finite one – could equivalently be set to zero without loss of generality after a proper renormalization of the income process (see Aiyagari 1994 or Açikgöz 2018 for a recent discussion). For instance, zero is the natural borrowing limit if productivity is null for one idiosyncratic state.[4]

We denote by $\mathcal{A}$ the σ-algebra of Borel sets of A. Similarly, we denote by $\mathcal{E}$ the power set of E.[5] We assume that the initial agents' distribution at date 0 is characterized by the measure μ_0 defined on the product σ-algebra $\mathcal{E} \times \mathcal{A}$, such that for any $A_0 \in \mathcal{A}$ and $E_0 \in \mathcal{E}$, $\mu_0(E_0, A_0)$ is the measure of agents with initial wealth in A_0 and history in E_0. Note that with a slight abuse of notation, we will denote $\mu_0(e_0, A_0)$, instead of $\mu_0(\{e_0\}, A_0)$, the measure of agents with initial income e_0 and initial wealth in A_0. The distribution of agents at date future dates $t \geq 1$ will depend on idiosyncratic history evolution but also on initial wealth – since, loosely speaking, the initial dependence at date 0 will propagate at "later" dates. Remarking that $\mathcal{E}^t$ is the σ-algebra defined on the product space E^t, we will denote by $\mu_t(E_t, A_0)$ the measure of agents with idiosyncratic history $e^t \in E_t$ at date t and initial wealth $a_0 \in A_0$. First, note that since the total measure of the population is

[3]See Section 4 and Appendix B for extensions of this simple framework, for instance to the case of an uncountable set E.

[4]The natural borrowing limit is the smaller borrowing limit (in absolute value) such that credit constraints do not bind in equilibrium. It is called the "present value" borrowing limit in Aiyagari (1994).

[5]Note that $\mathcal{E}$ can also be seen as the Borel sets of the discrete space E endowed with the discrete topology.

constant and equal to 1 at all dates, we have, at all dates t:

$$\int_{a_0 \in A} \sum_{e^t \in E^t} \mu_t(e^t, da_0) = 1.$$

Second, using Bayes' law, the measure μ_{t+1} can be expressed using the measure μ_t and transition probabilities $(\tilde{\Pi}_{e^t,e^{t+1}})_{e^t,e^{t+1}}$. Formally, for any $e^{t+1} \in E^{t+1}$ and any $A_0 \in \mathcal{A}$:

$$\mu_{t+1}(e^{t+1}, A_0) = \sum_{e^t \in E^t} \tilde{\Pi}_{e^t,e^{t+1}} \mu_t(e^t, A_0). \tag{1}$$

2.2 Agents' program

Agents are expected-utility maximizers with standard time-additive preferences. The discount factor $\beta \in (0, 1)$ is constant and the period utility function, denoted $u : \mathbb{R}_+ \to \mathbb{R}$, is twice continuously differentiable, increasing, and strictly concave.

Agents can transfer resources from one period to another through a security (capital shares) that pays off the deterministic interest rate r. However, as already explained, borrowing is limited and agents cannot borrow more than the amount $\underline{a} \geq 0$. Agents choose their consumption path $(c_t)_{t\geq 0}$ and their saving path $(a_{t+1})_{t\geq 0}$ so as to maximize the expected utility, subject to the credit limits and the borrowing constraint. The latter states that, at any date, spending in consumption and savings cannot exceed resources made of savings payoffs and income. Formally, the agent's program can be expressed as:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right), \tag{2}$$

$$\text{s.t. } c_t + a_{t+1} \leq (1 + r)\, a_t + e_t, \tag{3}$$

$$a_{t+1} \geq -\underline{a}, \tag{4}$$

$$a_0, e_0 \text{ given.} \tag{5}$$

In equation (2), the unconditional expectation $\mathbb{E}[\cdot]$ is taken over future income stream, which is the only stochastic variable. A solution to the households problem (2)–(5) is a sequence of measurable consumption functions $c_t : E^t \times A \to \mathbb{R}^+$, and a sequence of measurable Lagrange multipliers on the credit constraint $\nu_t : E^t \times A \to \mathbb{R}^+$, solving the standard Euler equation at all dates t:

$$u'\left(c_t(e^t, a_0)\right) = \beta\,(1 + r) \sum_{e^{t+1} \in E^{t+1}} \tilde{\Pi}_{e^t,e^{t+1}} u'\left(c_{t+1}(e^{t+1}, a_0)\right) + \nu_t(e^t, a_0), \tag{6}$$

where $\sum_{e^{t+1} \in E^{t+1}} \tilde{\Pi}_{e^t,e^{t+1}}[\cdot]$ is the conditional expectation operator written in explicit form. Note that when the credit constraint does not bind at date t for initial wealth a_0 and history e^t, we have $\nu_t(e^t, a_0) = 0$. The quantity $\nu_t(e^t, a_0)$ can be interpreted as the shadow price of the agent's credit constraint (4). The saving functions $a_{t+1} : E^t \times A \to \mathbb{R}^+$ can then be deduced from the budget constraints.

2.3 Production

In each period t, a representative firm produces output Y_t using capital K_t and labor L_t. The firms rents the capital at a rate r_t and labor at a wage w_t. Production net of depreciation is $F(K_t, L_t) - \delta K_t$ where $\delta \in (0,1)$ is the depreciation rate and $F(\cdot,\cdot)$ is a constant-returns-to-scale production function, strictly increasing and concave. Profit maximization implies (F_K and F_L denote the partial derivatives):

$$r_t = F_K(K_t, L_t) - \delta, \quad w_t = F_L(K_t, L_t). \tag{7}$$

Capital and total labor supply are defined by market clearing conditions:

$$K_t = \int_{a_0 \in A} \sum_{e^t \in E^t} a_t(e^t, da_0)\mu_t(e^t, da_0), \qquad L_t = \int_{a_0 \in A} \sum_{e^t \in E^t} e_t \mu_t(e^t, da_0). \tag{8}$$

2.4 Equilibrium

We now define the concept of stationary competitive equilibrium.

Definition 1 (Equilibrium) *A stationary competitive equilibrium is a collection of individual allocations $(c_t^i, a_t^i)_{t\geq 0, i \in \mathcal{I}}$, of aggregate quantities (K, L), of price processes (w, r), such that, for an initial wealth distribution $(a_0^i)_{i\in\mathcal{I}}$, we have:*

1. *given prices, individual allocations $(c_t^i, a_t^i)_{t\geq 0, i\in\mathcal{I}}$ solve the agent's program (2)–(5);*
2. *financial, labor, and goods markets clear at all dates. There exist $K, L \in \mathbb{R}$, such that for any $t \geq 0$:*

$$K = \int_{a_0 \in A} \sum_{e^t \in E^t} a_t(e^t, da_0)\mu_t(e^t, da_0), \qquad L = \int_{a_0 \in A} \sum_{e^t \in E^t} e_t \mu_t(e^t, da_0),$$
$$F(K, L) = \int_{a_0 \in A} \sum_{e^t \in E^t} c_t(e^t, da_0)\mu_t(e^t, da_0) + \delta K.$$

3. *factor prices are consistent with (7): $r = F_K(K, L) - \delta$ and $w = F_L(K, L)$.*
4. *The distribution of marginal utilities is constant (and finite) over time, i.e.:*

$$\int_{a_0 \in A} \sum_{e^t \in E^t} u'\left(c_t(e^t, da_0)\right)\mu_t(e^t, da_0) = \int_{a_0 \in A} \sum_{e^{t+1} \in E^{t+1}} u'\left(c_{t+1}(e^t, da_0)\right)\mu_{t+1}(e^t, da_0). \tag{9}$$

In our stationary equilibrium definition, points 1 to 3 are very standard – see Açikgöz (2018) for instance. We focus on the equilibrium with constant prices and constant aggregate quantities. Point 4 is however weaker than in the standard formulation, which usually assumes that the distribution of asset holdings is constant in the economy. We only require the aggregation of individual marginal utilities to be constant.

3 Main result

The following proposition contains our main result.

Proposition 1 (Interest rate) *In any existing stationary equilibrium, the interest rate must satisfy:*

$$\frac{1}{\beta} - (1+r) = \frac{\int_{a_0} \sum_{e^t \in E^t} \nu(e^t, a_0)\mu_t(e^t, da_0)}{\beta \int_{a_0} \sum_{e^t \in E^t} u'\left(c_t(e^t, a_0)\right)\mu_t(e^t, da_0)}. \tag{10}$$

As the proof is straightforward, we first provide it before discussing the Proposition.

Proof. Aggregating the Euler equations (6) over all possible histories $e^t \in E^t$ and all initial asset holdings $a_0 \in A$ yields:

$$\begin{aligned}
&\int_{a_0} \sum_{e^t \in E^t} u'\left(c_t(e^t, a_0)\right)\mu_t(e^t, da_0) - \int_{a_0} \sum_{e^t \in E^t} \nu(e^t, a_0)\mu_t(e^t, da_0) \\
&= \beta\,(1+r) \int_{a_0} \sum_{e^t \in E^t} \sum_{e^{t+1} \in E^{t+1}} \Pi_{e^t, e^{t+1}} u'\left(c_{t+1}(e^{t+1}, a_0)\right)\mu_t(e^t, da_0), \\
&= \beta\,(1+r) \int_{a_0} \sum_{e^{t+1} \in E^{t+1}} u'\left(c_{t+1}(e^{t+1}, a_0)\right)\left(\sum_{e^t \in E^t} \Pi_{e^t, e^{t+1}} \mu_t(e^t, da_0)\right),
\end{aligned}$$

where the last equality comes from the permutation of the two finite sums. Using the recursive definition of (μ_t) in equation (1) stating that the term between brackets is μ_{t+1}, our stationarity property (9) readily implies the expression (10). Indeed, note that we can rule out that $\int_{a_0} \sum_{e^t \in E^t} u'\left(c_t(e^t, a_0)\right)\mu_t(e^t, da_0) = 0$ for some t. Indeed, should it hold, this would imply $u'\left(c_t(e^t, a_0)\right) = 0$ almost surely and thus $c_t(e^t, a_0) = \infty$ almost surely (because $u' > 0$ and $u'' < 0$), which is not compatible with goods market clearing. ■

Proposition 1 states that the gap between the discount and the interest rates is proportional to average shadow price of credit constraints, where the average is computed over all possible idiosyncratic histories and initial asset holdings. This equilibrium outcome can be seen from two perspectives. First, if the interest rate $1 + r$ is below the discount rate $1/\beta$, then self-insurance is costly. As a consequence, households rationally choose not to perfectly self-insure themselves and hit the credit limit with probability 1 in some states of the world. This is an important step in the proof of existence of Açikgöz (2018), for instance. Conversely, when the credit constraint binds in some states of the world, households want to save to transfer resources to this state of the world. As a consequence, they accept a lower return, relative to the complete market economy, due to this self-insurance motive. This generates a liquidity premium on the asset.[6]

For the sake of simplicity, Proposition 1 and its proof are stated with a sequential formulation of the model. It also holds using a recursive formulation (see Appendix A).

[6] Liquidity is here defined has the ability of an asset to transfer some wealth in the state of the world where the credit constraint binds.

The following corollary is immediate.

Corollary 1 (Stationary equilibrium characterization) *Any existing stationary equilibrium must feature either:*

- $\beta(1+r) < 1$ *and binding credit constraints for a positive measure of agents; or*
- $\beta(1+r) = 1$ *and non-binding credit constraints (almost surely).*

Corollary 1 provides a straightforward characterization of any stationary equilibrium (whenever it exists).

We conclude this section by a very general impossibility result.

Corollary 2 (Existence) *If* $\text{Card}\, E \geq 2$ *and if for all* $e, e' \in E$, $\Pi_{ee'} \in (0,1)$, *there cannot exist a stationary equilibrium when credit constraints do not bind for a positive mass of agents.*

Proof. The corollary is the direct consequence of Proposition 1. It is proven by contradiction. If a stationary equilibrium exists and credit constraints do not bind, we must have $\beta(1+r) = 1$, which has been shown to be incompatible with the existence of a stationary equilibrium. More precisely, Chamberlain and Wilson (2000) (Corollary 2, p. 381) show under very general conditions that if the discounted income stream has sufficient variability, then consumption path diverges to infinity almost surely. In our our case, with a stationary Markovian process, the variability condition implies to have $\mathbb{V}_0\left[\sum_{t=0}^{\infty} \beta^t e_t | e_0\right]$ bounded away from 0 for any initial state e_0. Since $\beta > 0$, $\mathbb{V}_0\left[\sum_{t=0}^{\infty} \beta^t e_t | e_0\right] = 0$ implies either a unique income level (and thus no income risk) or a transition matrix $(\Pi_{e'e})$ containing an attractive state. The first point is made impossible by $\text{Card}\, E \geq 2$ and the second one by $\Pi_{ee'} \in (0,1)$ for all (e, e'). ∎

Corollary 1 states that any stationary equilibrium – whenever it exists – must feature binding credit constraint for a positive measure of agents, as long as income is sufficiently variable. In the case of the finite-state Markovian process, this only rules out polar cases: one-state Markov chain and conditionally deterministic Markov chains (i.e., with transition matrices containing only zeros and ones).

An example. For illustration purposes, we simulate a standard incomplete market economy. We plot the asset demand for varying credit limits. Agents are assumed to have a labor endowment equal to $e_1 = 1$ or $e_2 = 0.8$. The transition matrix across these two idiosyncratic states is assumed to be symmetrical.

$$\Pi = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}.$$

The discount factor is $\beta = 0.96$, the production function if $F(K, L) = K^{\alpha} L^{1-\alpha}$, with $\alpha = 1/3$ and the depreciation rate is $\delta = 0.1$. We then compute the asset demand of households as a function of the interest rate r. We perform this exercise in two environments. In the first one, the credit limit is set to $\underline{a} = 0$. In the second, the credit limit is close to the natural borrowing constraint $\underline{a}^n(r) = -(1-\varepsilon) we_1/(1+r)$, where $\varepsilon = 1\%$. In other words, the credit limits is 99% of the natural borrowing limit.

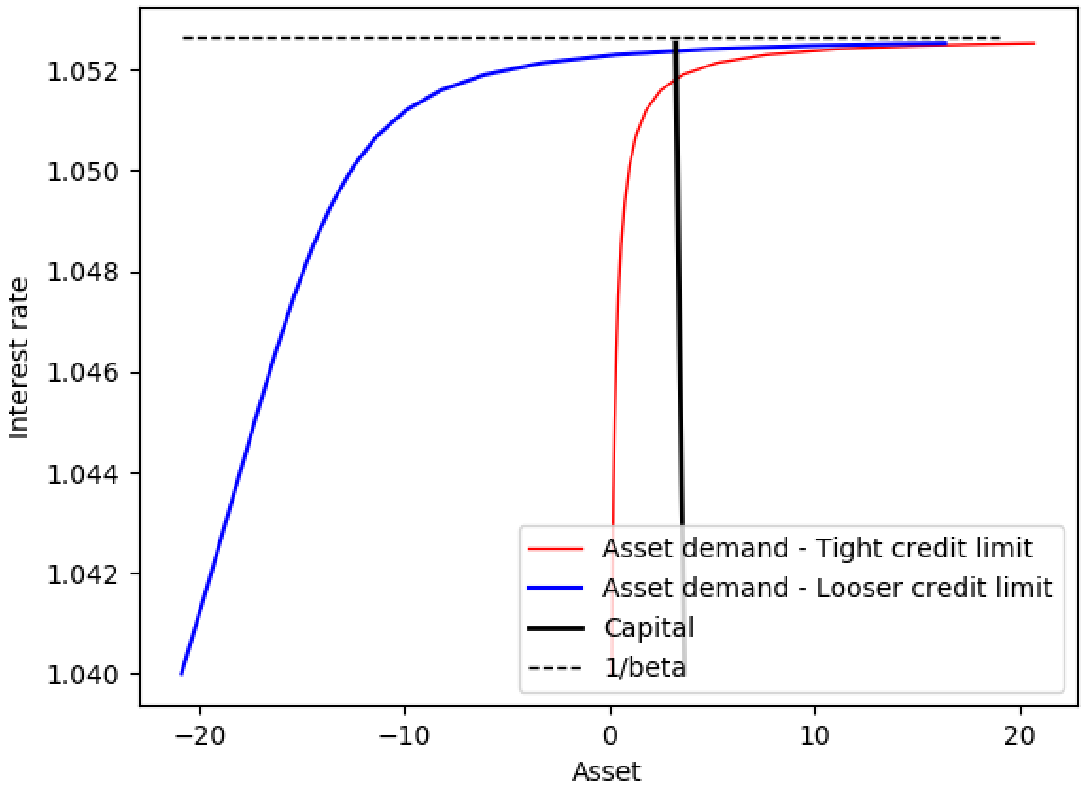

Figure 1: Asset supply and demand for different credit limits

Figure 1 plots the results, using the Aiyagari (1994) representation. The x-axis corresponds to the asset quantity of assets and the y-axis to the interest rate. The black solid line is the capital demand by the firm, which is steadily downward slopping (and looks vertical due to the scale of the x-axis). The horizontal black dashed line plots the interest rate $1 + r = 1/\beta$, which would prevail in the complete market economy. The red line plots the asset demand by households in the economy where $\underline{a} = 0$. The blue solid line plots the asset demand when the borrowing is set to $\underline{a}^n(r)$, which varies with the interest rate r.

A looser credit limit translates the asset demand to the left. This result is derived theoretically by Acemoglu and Jensen (2015). A tightening of the credit limit, which is a "positive shock" using their wording, increases the savings of households for self-insurance motive. The equilibrium interest rate in these two economies is at the intersection between the black solid line and the relevant asset-demand curve. The severity of the credit limit, measured as the right hand side of equation (10), is the difference between the equilibrium interest rate and the dashed-line representing $1/\beta$.

4 Extensions

In this section, we briefly discuss various extensions to the result of Proposition 1, as well as those of Corollaries 1 and 2 (besides the recursive formulation of Appendix A).

A continuous income space. Generalizing the income space E to a continuous space – while maintaining the Markovian structure – is rather straightforward and lets all our results unchanged. The proof is also roughly the same and the only major difference is that the discrete sum over idiosyncratic histories becomes an integral. See Appendix B for a formal presentation.[7]

The results, in particular Proposition 1 and Corollary 1, would also hold for much more general income process (including non-Markovian ones). As in Chamberlain and Wilson (2000), the non-existence result requires sufficient variability in the income process.

Capital tax. Proposition 1, and Corollaries 1 and 2 will still hold if one allows for a linear capital tax. The only difference is that results will be formulated in terms of post-tax interest rate (and not pre-tax one). The post-tax rate is the one that is faced by households and thus the rate that matters for households' decisions.

Endogenous labor supply. Introducing endogenous labor supply (with idiosyncratic productivity risk) will not affect the results of Proposition 1 and Corollary 1. Indeed, the individual Euler equations for consumption remain valid, even though they could also depend on labor choices. Their aggregation still lead to the characterization of the interest rate at any stationary equilibrium.

The existence result of Corollary 2 will however depend on the formalization of the labor supply. Marcet, Obiols-Homs, and Weil (2007) have shown that a stationary equilibrium can exist if $\beta(1+r) = 1$ if the wealth effect on the labor supply is high enough to reduce labor income and capital accumulation of wealth-rich agents. On the opposite, if the wealth effect is low or absent – as in the case of a Greenwood-Hercowitz-Huffman utility function – then no stationary equilibrium exists and Corollary 2 holds.

[7]Obviously, the σ-algebra $\mathcal{E}$ on E is now the Borel algebra of E and not the power set of E. The measure μ_t defined on $\mathcal{E}^t \times \mathcal{A}$ is thus changed accordingly.

Summary. In short, Proposition 1 and Corollary 1 are robust to various extensions and will hold as long as the two following key ingredients are present.

1. There should exist one asset whose interest rate is not affected by the agents' idiosyncratic risk. In loose terms, it needs to be "taken out" of the expectation for their idiosyncratic risk.

2. The aggregation of individual Euler equations should lead to the aggregate values of marginal utilities over the whole population. This notably implies the existence of a stationary distribution and also that in presence of mortality (as in the Blanchard-Yaari model), some specific assumptions about the initial endowments of new-born agents should be made to maintain our result.

5 Implications for the literature

The previous analysis has important implications for the literature. First, the seminal paper of Aiyagari (1994) focuses on stationary equilibria in various contexts, including the case of non-binding credit-constraints. In some analysis, the borrowing limit is then set to a lower value than what is called the "present value" borrowing limit, or the natural borrowing limit. One of the conclusions of the analysis is that in this environment, Ricardian equivalence holds. Unfortunately, the stationary equilibrium does not exist and the long-run effect of public debt should be analyzed with binding credit constraints.

Second, Aiyagari (1995) analyzes optimal capital and labor taxes, and public debt in a standard incomplete-market model, where the government has to finance public consumption. Aiyagari finds that, in this economy, the government sets the before-tax interest rate $\tilde{r}$ equal to the discount rate $1/\beta - 1$, or equivalently: $\beta(1 + \tilde{r}) = 1$. This is called the "Golden Rule". As no stationary equilibrium can exist when $\beta(1 + r) = 1$ (r denoting again the post-tax interest rate faced by agents), he concludes that capital taxes have to be positive for any stationary equilibrium to exist. Our finding confirms this statement. More precisely, our previous analysis shows that the government implicitly chooses the "measure" of agents who are credit-constrained. This property could help prove that stationary equilibria with optimal positive capital tax indeed exist, what has not been achieved yet, to the best of our knowledge.

Third, the seminal papers of Angeletos and Calvet (2005, 2006) consider non-binding credit constraints in the CARA-normal case in an economy where households are entrepreneurs endowed with their own production function and facing an idiosyncratic production risk (either through the total productivity factor or the depreciation rate). Their results are derived in a non-stationary equilibrium, where the variance of consumption is unbounded. Our non-existence result, which applies to their framework, confirms that focusing on a stationary equilibrium is not possible in such setups.

Finally, a recent literature is analyzing optimal Ramsey policies in incomplete market economies. A potential method to derive first-order conditions of the planner is the "primal approach", which is often used in complete market economies (see Chari, Nicolini, and Teles 2019 for a recent discussion and a formulation). This approach uses the non-binding Euler equations of households to substitute for the real interest rate. As shown by our analysis, no stationary equilibrium exists in this case, what limits the scope of this approach to non-stationary economies. An alternative strategy is to use the "Lagrangian approach" developed by Marcet and Marimon (2019). Some progress in this direction has been made by Açikgöz, Hagedorn, Holter, and Wang (2018) and Le Grand and Ragot (2019).

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Appendix

A The proof using a recursive representation

The setup and the proofs of the main paper were provided using a sequential representation for the sake of clarity. We prove here that results still hold using a recursive representation. As we provide a characterization of the interest rate and existence in any stationary equilibrium, we will simply assume that it exists, and derive proofs by contradiction. The framework is the same as the one described in Section 2 – in particular the income space E is discrete.

We assume that there exists a stationary probability measure $\psi : \mathcal{E} \otimes \mathcal{A} \rightarrow [0,1]$ such that $\psi\left(e, A_0\right)$ is the stationary measure of agents of productivity type $e \in E$, who hold a quantity of assets in the set $A_0 \subset A$. The program of the agents written in recursive form is defined by the value function $V\left(e,a\right) = max_{c,a'} u\left(c\right) + \beta \mathbb{E} V\left(e', a'\right)$, subject to the budget constraint $a' + c = a\left(1+r\right) + we$ and to the credit constraint $a' \geq -\underline{a}$.

A stationary equilibrium is defined as a set of policy functions $c\left(e,a\right)$ and $a' = g_a\left(e,a\right)$, for consumption and savings respectively, prices r, w, and a stationary distribution ψ such that: (i) the policy functions solve the agents program of agents when prices are given; (ii) capital and labor markets clear: $K = \sum_{e \in E} \int_A g_a\left(e,a\right) d\psi\left(e, da\right)$ and $L = \sum_{e \in E} \int_A e d\psi\left(e, da\right)$; (iii) ψ is invariant by the transition functions generated by the policy rules and the law of motions of the income space. Formally, for all $A_0 \in \mathcal{A}$ and $e \in E$:

$$\psi\left(e, A_0\right) = \sum_{\tilde{e} \in E} \int_{a \in A} 1_{A_0}\left(g_a(a, \tilde{e})\right) \Pi_{\tilde{e},e} \psi\left(\tilde{e}, da\right),$$

where $1_{A_0}\left(g_a(a,\tilde{e})\right) = 1$ if $g_a(a,\tilde{e}) \in A_0$ and 0 otherwise. The first order condition for the agent's program can be written as:

$$u'\left(c(e,a)\right) = \beta\left(1+r\right) \sum_{e' \in E} \Pi_{e,e'} u'\left(c\left(e', g_a(e,a)\right)\right) + \nu\left(e,a\right), \tag{11}$$

where $\nu\left(e,a\right)$ is the Lagrange multiplier on the credit constraint.

In such equilibrium, the distribution of marginal utilities is constant over time, and so is the average marginal utility in the economy. Seen from the current period, the next period average marginal utility is given by the policy rules and the law of motion for the productivity shock. Indeed, for agents having a productivity level e and a wealth level $a \in A$, their next period marginal utility if they happen to have productivity $e' \in E$ is given by $u'\left(c\left(e', a'\right)\right) = u'\left(c(e', g_a(a,e), e')\right)$ and there will be a fraction $\Pi_{e,e'} \times \psi\left(da, e\right)$ of such agents. As a consequence, the next period average marginal utility in the economy being equal to the current period one, we have:

$$\sum_{e\in E}\sum_{e'\in E}\int_A u'\left(c\left(e',g_a\left(e,a\right)\right)\right)\Pi_{e,e'}\psi\left(e,da\right)=\sum_{e\in E}\int_A u'\left(c\left(e,a\right)\right)\psi\left(e,da\right). \tag{12}$$

Integrating the Euler equation (11), we deduce

$$\sum_{e\in E}\int_A u'\left(c\left(e,a\right)\right)\psi\left(e,da\right)=\beta\left(1+r\right)\sum_{e\in E}\int_A \sum\nolimits_{e'\in E}\Pi_{e,e'}u'\left(c\left(e',g_a\left(a,e\right)\right)\right)\psi\left(e,da\right)$$
$$+\sum_{e\in E}\int_A \nu\left(e,a\right)\psi\left(e,da\right).$$

Using the stationarity of the average marginal utility in the economy of equation (12), we can directly state the following proposition.

Proposition 2 (Interest rate) *If a recursive stationary equilibrium exists, then:*

$$1+r=\frac{1}{\beta}-\frac{\sum_{e\in E}\int_A\nu\left(e,a\right)\psi\left(e,da\right)}{\beta\sum_{e\in E}\int_A u'\left(c\left(e,a\right)\right)\psi\left(e,da\right)}.$$

Proposition 2 is the strict parallel of Proposition 1 for the recursive formulation. As a consequence, when credit constraints do not bin, we again have $\beta(1+r)=1$ for any existing stationary equilibrium, and Corollaries 1 and 2 still hold.

B Proof for a continuous income state-space

We now provide the proof for a continuous income space $E\subset\mathbb{R}^+$ and its Borel algebra $\mathcal{E}$. We consider a transition kernel p, which extends the notion of transition probabilities. More precisely, for any $e\in E$ and any $E_0\in\mathcal{E}$, $p(E_0|e)$ is the probability to reach an income $e'\in E_0$ from the income e. We can deduce from p the transition kernel for histories, denoted $\tilde{p}$, that is defined by: $\tilde{p}\left(\prod_{\tau=0}^{t+1}E_\tau\,|e^t\right)=\int_{e^{t+1}\in\prod_{\tau=0}^{t+1}E_\tau}1_{e^{t+1}\succeq e^t}p(de_{t+1}|e_t)$ for all $\prod_{\tau=0}^{t+1}E_\tau\in\mathcal{E}^{t+1}$ and all $e^t\in E^t$. Note that making the relationships between e_t and e^t explicit ($e^t=(\tilde{e}^{t-1},e_t)$) and between e_{t+1} and e^{t+1} explicit ($e^{t+1}=(\tilde{e}^t,e_{t+1})$), the definition of $\tilde{p}$ can also be written as: $\tilde{p}\left(\prod_{\tau=0}^{t+1}E_\tau\,|(\tilde{e}^{t-1},e_t)\right)=\int_{(\tilde{e}^t,e_{t+1})\in\prod_{\tau=0}^{t+1}E_\tau}1_{\tilde{e}^t=e^t}p(de_{t+1}|e_t)$.

As in the main text, we start from an initial distribution μ_0 defined over $\mathcal{E}\times\mathcal{A}$, such that $\int_{E\times A}\mu_0(de_0,da_0)=1$. The distribution at date t, denoted μ_t is defined over $\mathcal{E}^t\times\mathcal{A}$ and verifies the following recursion:

$$\mu_{t+1}\left(\prod_{\tau=0}^{t+1}E_\tau,A_0\right)=\int_{(e^{t+1},a_0)\in\prod_{\tau=0}^{t+1}E_\tau\times A_0}\int_{e^t\in E^t}\tilde{p}(de^{t+1}|e^t)\mu_t(de^t,da_0), \tag{13}$$

where $E_\tau\in\mathcal{E}$ and $A_0\in\mathcal{A}$. Using infinitesimal notation, we can alternatively write μ_{t+1}:

$$\mu_{t+1}(de^{t+1},da_0)=\int_{e^t\in E^t}\tilde{p}(de^{t+1}|e^t)\mu_t(de^t,da_0). \tag{14}$$

We can now state a result similar to the one of Proposition 1.

Proposition 3 (Interest rate) *In any existing stationary equilibrium, the interest rate has to satisfy:*

$$1 + r = \frac{1}{\beta} - \frac{\int_{E^t \times A} \nu(e^t, a_0) \mu_t(de^t, da_0)}{\beta \int_{E^t \times A} u'\left(c_t(e^t, a_0)\right) \mu_t(de^t, da_0)}. \tag{15}$$

Proof. The Euler equation of the agent's program can be written as:

$$u'\left(c_t(e^t, a_0)\right) = \beta\,(1 + r) \int_{e^{t+1} \in E^{t+1}} u'\left(c_{t+1}(e^{t+1}, a_0)\right) \tilde{p}(de^{t+1}|e^t) + \nu(e^t, a_0) \tag{16}$$

Integrating over the whole population, the set of Euler equations (16) over the whole distribution of agents yields:

$$\int_{(e^t, a_0) \in E^t \times A} u'\left(c_t(e^t, a_0)\right) \mu_t(de^t, da_0) = \int_{(e^t, a_0) \in E^t \times A} \nu(e^t, a_0) \mu_t(de^t, da_0)$$
$$\beta\,(1 + r) \int_{(e^t, a_0) \in E^t \times A} \int_{e^{t+1} \in E^{t+1}} u'\left(c_{t+1}(e^{t+1}, a_0)\right) \tilde{p}(de^{t+1}|e^t) \mu_t(de^t, da_0).$$

Since $(e^{t+1}, a_0) \mapsto u'\left(c_{t+1}(e^{t+1}, a_0)\right)$ is assumed to be Lebesgue integrable, the Fubini's theorem yields:

$$\int_{(e^t, a_0) \in E^t \times A} u'\left(c_t(e^t, a_0)\right) \mu_t(de^t, da_0) = \int_{(e^t, a_0) \in E^t \times A} \nu(e^t, a_0) \mu_t(de^t, da_0)$$
$$\beta\,(1 + r) \int_{(e^{t+1}, a_0) \in E^{t+1} \times A} u'\left(c_{t+1}(e^{t+1}, a_0)\right) \int_{e^t \in E^t} \tilde{p}(de^{t+1}|e^t) \mu_t(de^t, da_0),$$

or using the recursive definition (14) of μ_{t+1}:

$$\int_{E^t \times A} u'\left(c_t(e^t, a_0)\right) \mu_t(de^t, da_0) = \beta\,(1 + r) \int_{E^{t+1} \times A} u'\left(c_{t+1}(e^{t+1}, a_0)\right) \mu_{t+1}(de^{t+1}, da_0)$$
$$+ \int_{E^t \times A} \nu(e^t, a_0) \mu_t(de^t, da_0)$$

Then using stationarity, stated similarly to equation (9), implies equality (15). ■

Loosely speaking, introducing a continuous state-space only changes the integral, which is now a continuous sum over histories, while it is a discrete sum in the main text. Proposition 3 is thus very similar to Proposition 1, and Corollaries 1 and 2 still hold.