Should we increase or decrease public debt? Optimal fiscal policy with heterogeneous agents*

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Abstract

We analyze optimal fiscal policy in a heterogeneous-agent model with capital accumulation and aggregate shocks, where the government uses public debt, capital tax and progressive labor tax to finance public spending. First, he existence of a steady-state equilibrium is proven to depend on three conditions, which have different economic interpretations: a Laffer condition, a Blanchard-Kahn condition and a Straub-Werning condition. First, the equilibrium can feature both a positive level of public debt and capital tax at the steady state, to correct for non-optimal private saving. Second, the optimal public debt increases after a positive public spending shock when its persistence is low, whereas it decreases when its persistence is high, due to a tradeoff between consumption smoothing and the reduction of distortions. We show that our results hold in a quantitative heterogeneous-agent model, where the optimal dynamics of the whole set of fiscal tools is analyzed. The general model also provides new results on optimal tax progressivity and the dynamics of labor tax.

Keywords: Heterogeneous agents, optimal fiscal policy, public debt

JEL codes: E21, E44, D91, D31.

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1 Introduction

What is the optimal level of public debt? Should it increase or decrease when public spending is increasing? After a positive public spending shock, should the government increase temporarily capital tax or other distorting taxes, affecting the progressivity of the tax system? These old questions are likely to stay relevant in the coming years in many countries. Such questions require considering both distorting and redistributive effects of tax changes, and short and long-run implications of debt and tax dynamics, considering general equilibrium effects. Heterogeneous-agent models in the tradition of as in the Bewley-Huggett-Imohoroglu-Aiyagari Bewley, 1983; Imrohoroğlu, 1989; Huggett, 1993; Aiyagari, 1994; Krusell and Smith, 1998) are relevant tools to analyze optimal fiscal policy as they generate realistic amount of heterogeneity together with all general and dynamic equilibrium effects. However, after some seminal investigations of optimal fiscal policy in these environements (Aiyagari (1995) and Aiyagari and McGrattan (1998)), the literature has moved toward positive analysis (such as Floden (2001)Heathcote (2005) or Rohrs and Winter (2017) among many others) or normative policy in steady-state environements (Conesa et al. (2009)) due to methodological difficulties to solve for optimal policy in heterogeneous-agent models with aggregate shocks.

This paper analyses optimal policy in heterogeneous-agent models, considering capital accumulation, progressive labor income taxation as in Heathcote et al. (2017), capital tax and public debt and aggregate shocks. The only friction considered is incomplete market for idiosyncratic risk and credit constraints (which appear to be the main friction), and the limited set of fiscal instruments which could be used to provide insurance. In particular, the planner can't use lump-sum taxes, which is known to to possibly restore Ricardian equivalence in some environments (Bhandari et al. (2017)). This analysis admittedly abstracts from other frictions, such an nominal rigidities or frictional labor markets, to first identify mechanisms which will be present in over environments. Considering capital accumulation appears to be important to endogenously produce store of value in possibly unlimited amount and to additionally characterize the optimal level and dynamics of capital tax. In such models, we study the optimal dynamics of fiscal tools after an increase in public spending, considering as a benchmark a Ramsey program with commitments.

In our model, the optimal level of public debt is well defined. Although public debt is "liquidity" in the sense of Woodford (1990) (that is a public store of value), its main function is a saving "absorber", as the private saving is too high when credit constraints are binding. This absorption is costly as the planner has to raise resources with distorting taxes for interest payment on public debt. The role of capital tax and labor tax is to raise resources to finance public spending.and the planner can choose both positive public debt and capital taxes. These first sets of result confirm Aiyagari (1995) claim, which has been challenged recently (Chen et al. (2021) in a special case and Chen et al. (2017)). The new set of results concerns the dynamics of fiscal instruments. We show that when the persistence of a positive fiscal shocks is low, public debt increases and taxes and the progressivity of income tax decrease in the first quarters. This implements consumption smoothing, as the needs for additional public resources are temporary. When the persistence is high, there is along-lasting needs for additional public resources, public debt decreases and taxes and progressivity increase in the first quarter to front-load the adjustment. As a consequence, whatever the size of public spending shock its persistence is a key driver of the optimal dynamics of public debt and tax progressivity in an economy where agents face credit constraints.

Solving optimal policies in heterogeneous-agent models with aggregate shocks is difficult, and we thus perform the analysis in two steps. First, we consider a special case of the general model to allow for analytical results for both steady-state value of the instruments and for the dynamics. The simplification is based on the assumption of deterministic income fluctuations as Woodford (1990) and a utility function without wealth effect for the labor supply (or GHH utility function following Greenwood et al. (1988) and Diamond (1998)). hence, the only friction is credit constraint. In this environment, we show that a long-run steady-state equilibrium exists with positive capital tax and possibly positive public debt under three conditions: : A Laffer condition, a Straub-Werning condition, and a Blanchard Kahn condition. These three conditions have different economic interpretations: The Laffer condition states that public spending should not be too high, otherwise distorting taxes cannot levy enough resources. The Straub-Werning condition, elaborating on Straub and Werning (2020) states that the public spending must be low enough, otherwise the planner wants to deviate from the steady state by continuously decreasing the capital stock (although it could levy enough resources at the steady state). The Blanchard-Kahn condition is a stability condition, which requires the planner not to deviate permanently from the steady state due to diverging public resources. When public spending is high the optimal fiscal system can exhibit both positive capital tax to tradeoff distortions with labor tax, and a positive public debt to absorb excess saving at the optimal tax system.

In the second step, we show that the dynamic properties of the fiscal system identified in the tractable model are valid in a quantitative heterogeneous-agents model. It is known that the tax system depends crucially on the social welfare function considered by the planner. As we are interest in the dynamics of the fiscal system, we first estimate a social welfare function consistent with the US tax system, following the follow *invert optimal approach,* used recently Heathcote and Tsujiyama (2021). Starting from a steady-state when then compute the optimal capital tax, labor tax progressivity and public debt after public spending shocks with different persistence. We indeed find that the results of the simple model are valid in this more general setup. Additionally, the general model shows that using a general utility function is important to characterize the dynamic of labor tax. For general utility functions the labor tax falls when the persistence is low, whereas the labor tax always increase, whatever the persistence, for the special case of GHH utility function.

Solving optimal policies is one frontier of heterogeneous-agent models. We here follow the methodology of LeGrand and Ragot (2022), which uses a Lagrangian approach to derive the policy of the planner. This approach allows for occasionnaly-binding credit constraints, which appear to be the key friction. The simulation of the model is based on the truncation approach of LeGrand and Ragot (2022), which shown to be accurate in Le Grand and Ragot, 2022. It is also used in LeGrand et al., 2021, for the analysis of monetary policy.

The paper is related to recent and relatively thin, quantitative literature studying optimal Ramsey policies in heterogeneous-agents models considering transitions (e.g., Conesa et al., 2009, Açikgöz et al., 2018, Dyrda and Pedroni, 2018, Nuño and Thomas, 2020, Bhandari et al., 2020). In this literature, LeGrand and Ragot (2022) use a Lagrangian approach (taken from Marcet and Marimon, 2019) and a truncation procedure to simulate the model. The gain is to allow for economic interpretation of the first-order condition of the planner.

The paper is also related to the literature on optimal capital taxation (Chari et al., 1994, Farhi, 2010, Chari et al., 2016, or Straub and Werning, 2020 among other), optimal tax redistribution and progressivity (e.g., Bassetto, 2014 or Heathcote et al., 2017). Where here consider the implications of occasionnaly-binding credit constraints.

The paper is also related to tractable models, allowing to derive optimal policies, (Bilbiie, 2008, Gottardi et al., 2014 Heathcote et al., 2017, Bilbiie and Ragot, 2020, Acharya et al., 2020, Heathcote and Tsujiyama, 2021 among many others). In this literature, we find that the framework of Woodford (1990) is particularly useful to study optimal fiscal policy.

The rest of the paper is organized as follows. In Section 2, we present the general environment. We present simplifying assumption and solve the tractable model in Section 3. The general model is then analyzed in Section 4 ans simulated in Section 5. Section 6 concludes.

2 The environment

Time is discrete and indexed by $t = 0, 1, \ldots$ The economy is populated by a continuum of agents distributed along a set I with measure ℓ. We follow Green (1994) and assume that the law of large numbers holds. The economy features production and a benevolent government that raises distorting taxes to finance exogenous public spending.

2.1 Risks

The economy is plagued by an idiosyncratic risk. The aggregate shock solely affects public spending, denoted by $(G_t)_{t\geq 0}$ and is therefore assimilated to public spending shocks. Furthermore, we assume that the whole path of public spending $(G_t)_{t\geq 0}$ becomes known to all agents in period 0. We will solve for the optimal adjustment of economy after such a shock, also known as a MIT shock.[1]

Agents face an uninsurable productivity risk, denoted by y. Individual productivity levels follow independent first-order Markov chains, whose state-space is the finite set $\{y_1, \ldots, y_K\}$ and the transition matrix is denoted by Π. We assume that the Markov chain admits a stationary distribution that is denoted by the K-dimensional vector n^y, verifying $n^y = \Pi n^y$.[2] When an agent is endowed with productivity y, she will earn a before-tax labor wage $\tilde{w}yl$, where l denotes her labor supply and $\tilde{w}$ is the before-tax hourly wage. At period t,the productivity of agent i is y_t^i, whereas the whole histoty of shcks up to period t is denoted $y^{i,t} \equiv \{y_0^i, ..., y_t^i\}$.

Finally and to simplify notatons, it is assumed that agents enter the economy at period with an itial distribution of wealth and productivity (a_{-1}, y) taken from an itial distribution Λ_0.

2.2 Production

The production sector is standard. The unique consumption good of the economy is produced by a profit-maximizing representative firm. At any date t, the firm production function combines labor L_t and capital K_{t-1} – that needs to be installed one period in advance – to produce Y_t units of the consumption good. The production function is assumed to be of the Cobb-Douglas type featuring constant returns to scale and capital depreciation. The TFP is normalized to one. Formally, the production is defined as:

$$Y_t = F(K_{t-1}, L_t) = K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1},$$

where $\alpha \in (0, 1)$ is the capital share and $\delta \in (0, 1)$ the capital depreciation rate.

The firm rents labor and capital at respective factor prices $\tilde{w}_t$ and $\tilde{r}_t$. The profit maximization conditions of the firm implies the following expressions for factor prices:

$$\tilde{w}_t = F_L(K_{t-1}, L_t) \text{ and } \tilde{r}_t = F_K(K_{t-1}, L_t). \tag{1}$$

2.3 Assets

In addition to capital, the economy also features public debt, whose size is denoted by B_t in period t. Public debt consists of one-period bonds issued by a benevolent government, that are assumed default-free. As there is not aggregate risk in this economy, both capital and public debt are substitue and must have the same after-tax return. Agenst face a credit limit and cannot borrow more than the exogenous amount $\overline{a}$.

[1]It is known that one can derive a first-order approximation of the dynamics of the model in the presence of aggregate shocks, using the information obtained from MIT shocks (Boppart et al., 2018, Auclert et al., 2019)

[2]In the quantitative analysis of Section 5, the Markov chain can be shown to be irreducible and aperiodic – hence n^y exists and is unique.

2.4 Government

A benevolent government has to finance the exogenous stream of public spending $(G_t)_{t\geq 0}$, by levying distorting taxes on capital and labor and issuing public debt. The tax on capital is linear, with a rate $(\tau_t^K)_{t\geq 0}$. The tax on labor income is assumed to be non-linear, and possibly time-varying. We denote by $T_t(\tilde{w}yl)$ the amount of labor tax paid by an agent earning the labor income $\tilde{w}yl$ by supplying l hours at a wage rate $\tilde{w}$ and a productivity y. We follow Heathcote et al. (2017) (henceforth, HSV) and consider the following functional form:

$$T_t(\tilde{w}yl) := \tilde{w}yl - \kappa_t(\tilde{w}yl)^{1-\tau_t}, \tag{2}$$

where κ captures the level of labor taxation and τ the progressivity. Both parameters are assumed to be time-varying and will be planner's instruments. When $\tau_t = 0$, labor tax is linear with rate $1 - \kappa_t$. Oppositely, the case $\tau_t = 1$ corresponds to full income redistribution, where all agents earn the same post-tax income κ_t. Functional form (2), combined with the linear capital tax, allows one to realistically reproduce the actual US system and its progressivity (see Heathcote et al. (2017) or Ferriere and Navarro, 2020).[3]

Using the public debt description of Section 2.3, the government budget constraint can thus be written as:

$$G_t + (1 + \tilde{r}_t)B_{t-1} = \int T_t(\tilde{w}_t y^i l_t^i)\ell(di) + \tau_t^K \tilde{r}_t(B_{t-1} + K_{t-1}) + B_t. \tag{3}$$

To simplify the government budget constraint, we introduce in the spirit of Chamley (1986), generalized post-tax factor prices, that are denoted without a tilde. We define the gross and net interest rates r_t and R_t, as well as the wage rate w_t, as follows:

$$w_t := \kappa_t(\tilde{w}_t)^{1-\tau_t}, \tag{4}$$

$$R_t := 1 + r_t = 1 + (1 - \tau_t^K)\tilde{r}_t. \tag{5}$$

The model can analytically be expressed using the pair of post-tax rates (R_t, w_t) rather than pre-tax ones $(\tilde{r}_t, \tilde{w}_t)$. This considerably simplifies the model exposition and its tractability. The values of the fiscal instruments τ_t^K, κ_t, and τ_t can then be recovered from the allocation.

With the post-tax notation and taking advantage of the property of homogeneity of the production function, we deduce that the governmental budget constraint (3) can also be written as follows:

$$G_t + R_t B_{t-1} + (R_t - 1)K_{t-1} + w_t \int_i (y_t^i l_t^i)^{1-\tau_t}\ell(di) = F(K_{t-1}, L_t) + B_t. \tag{6}$$

[3]The literature uses either the combination of a linear tax and of a lump-sum transfer (e.g., Dyrda and Pedroni, 2018, Açikgöz et al., 2018) or the HSV structure. Heathcote and Tsujiyama (2021) show that the HSV structure is quantitatively more relevant. Opting for the HSV tax structure enables us to discuss t he dynamics of optimal tax progressivity, following a public spending shock.

2.5 Agents' program and resource constraints

At each date t, agents consume a unique good in quantity c_t and supply labor in quantity l_t. They derive an instantaneous utility from consumption and labor supply denoted by $U(c_t, l_t)$. The utility function will be specified later on. Agents are expected utility maximizers with standard additive intertemporal preferences. The discount factor is constant and denoted $\beta \in (0, 1)$. Agents maximize at date 0 the expected discounted value of future utilities, equal to $\mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t U(c_t, l_t)\right]$, where $\mathbb{E}_0$ is the unconditional expectation over the aggregate risk and over the agent's own idiosyncratic risk.

When choosing their plans for consumption $(c_t)_{t\geq 0}$, labor supply $(l_t)_{t\geq 0}$, and savings $(a_t)_{t\geq 0}$ to maximize their expected utility, agents face two constraints: (i) a budget constraint, and (ii) a credit constraint. Their budget constraints state that agents' consumption and savings should be solely financed out of net labor income and net capital income. Using the post-tax rate definition (4), the post-tax labor income amounts to $\tilde{w}_t y_t^i l_t^i - T_t(\tilde{w}_t y_t^i l_t^i) = w_t(y_t^i l_t^i)^{1-\tau_t}$ for an agent supplying labor l_t^i with productivity y_t^i. The post-tax capital income is equal to $R_t a_{t-1}^i$ for an agent with beginning-of-period wealth a_{t-1}^i. Formally, the program of an agent i can be expressed as:

$$\max_{\{c_t^i, l_t^i, a_t^i\}_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t^i, l_t^i), \tag{7}$$

$$c_t^i + a_t^i = R_t a_{t-1}^i + w_t(y_t^i l_t^i)^{1-\tau_t}, \tag{8}$$

$$a_t^i \geq -\underline{a},\ c_t^i > 0,\ l_t^i > 0, a_{-1} \text{ given.} \tag{9}$$

The solution of the previous program is a set of policy rules $c_t : \mathcal{Y}^t \times \mathbb{R} \to \mathbb{R}^+$, $a_t : \mathcal{Y}^t \times \mathbb{R} \to [-\bar{a}; +\infty[$ and $l_t : \mathcal{Y}^t \times \mathbb{R} \to \mathbb{R}^+$ which determine consumption, saving and labour spully decisions as a function of the idiosyncratic history y_i^t of agent i, her initial wealth a_{-1}^i. However, to simplify the notation, we will simply write c_t^i, a_t^i and l_t^i (instead of $c_t(y_i^t, a_{-1}^i)$, $a_t(y_i^t, a_{-1}^i, Z^t)$ and $l_t(y_i^t, a_{-1}^i, Z^t)$). We will use the same notation for all variables, as summarized by the next remark.

Remark 1 (Simplifying Notation) *If an agent has an idiosyncratic history y_i^t, and initial wealth a_{-1}^i at period t, we will then denote by X_t^i the realization in state (y_i^t, a_{-1}^i) of any random variable $X_t : \mathcal{Y}^t \times \mathbb{R} \to \mathbb{R}$*

A consequence of Remark 1 is that the aggregation of the variable X_t in period t over the whole agent population will be written as $\int_i X_t^i \ell(di)$, instead of the more involved explicit notation $\int_{a_{-1}} \sum_{y^t \in \mathcal{Y}^t} \theta_t(y^t) X(y^t, a_{-1}) d\Lambda_0(a_{-1}, y_0)$.

Denoting by $\beta^t \nu_t^i \geq 0$ the Lagrange multiplier on the agent's credit constraint, the consumption

Euler equation can be written as:

$$U_c(c_t^i, l_t^i) = \beta \mathbb{E}_t \left[R_t U_c(c_{t+1}^i, l_{t+1}^i) \right] + \nu_t^i, \tag{10}$$

where U_c and U_l denote the derivatives of U with respect to the first and second variables, respectively. Note that, because of our assumption of MIT shocks, the expectation operator in (10) as well as in the rest solely concerns idiosyncratic shocks.

The labor Euler equation yields:

$$-U_l(c_t^i, l_t^i) = (1 - \tau_t) w_t (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i). \tag{11}$$

The clearing of financial and labor markets implies the following relationships:

$$A_t = K_t + B_t \text{ and } \int y_t^i l_t^i \ell(di) = L_t. \tag{12}$$

The clearing of the goods market reflects that the sum of aggregate consumption, public spending and new capital stock balances the output production and past capital:

$$\int_i c_t^i \ell(di) + G_t + K_t = K_{t-1} + F(K_{t-1}, L_t). \tag{13}$$

We can now formulate our definition of a sequential equilibrium in this economy.

Definition 1 (Competitive equilibrium) *A competitive equilibrium is a collection of individual variables $(c_t^i, l_t^i, a_t^i)_{t \geq 0, i \in \mathcal{I}}$, of aggregate quantities $(K_t, L_t, Y_t)_{t \geq 0}$, of price processes $(\tilde{w}_t, \tilde{r}_t)_{t \geq 0}$, of fiscal policy $(\tau_t^K, \kappa_t, \tau_t, B_t)_{t \geq 0}$ and of public spending $(G_t)_{t \geq 0}$ such that, for an initial distribution of wealth and productivity $(a_{-1}^i, y_0^i)_{i \in \mathcal{I}}$, and for initial values of the aggregate shock z_0 and of capital stock and public debt verifying $K_{-1} + B_{-1} = \int_i a_{-1}^i \ell(di)$, we have:*

1. *given prices, individual strategies $(c_t^i, l_t^i, a_t^i)_{t \geq 0, i \in \mathcal{I}}$ solve the agent's optimization program in equations (7)–(9);*

2. *financial, labor, and goods markets clear: for any $t \geq 0$, equations (12) and (13) hold;*

3. *the government budget is balanced: equation (3) holds for all $t \geq 0$;*

4. *pre-tax factor prices $(\tilde{w}_t, \tilde{r}_t)_{t \geq 0}$ are consistent with the firm's program (1);*

A stationary equilibirum is an equilibrium where all aggregate variables are constant.

2.6 The Ramsey equilibrium

The first step to consider optimal policy is to define a Social Welfare Function. We assume that the planner considers a weighted sum of agents' utilities, where the sum depends on their productivity at date t: $\omega(y_t^i)$, as in Heathcote and Tsujiyama (2021). This general specification nest the utilitarian Social Welfare Function, used in Section 3, but also more general Social Welfare Functions; to reproduce the US fiscal system in the quantitative Section 5. Formally, the planner's aggregate welfare criterion can be expressed[4] as:

$$W_0 = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \int_i \omega(y_t^i) U(c_t^i, l_t^i) \ell(di) \right]. \tag{14}$$

The Ramsey program consists in finding the fiscal policy that corresponds to the competitive equilibrium with the highest aggregate welfare for the given Social Welfare function. This problem is difficult. The labor tax directly affects the labor supply, the capital tax directly affects the saving incentives, public debt directly affects the capital stock for a given total private saving. All these instruments have indirect general equilibrium effecst on prices and thus on the welfare of heterogeneous agents. As mentioned in the introduction, the existence of stationary equilibria with strictly positive values for the instrument is an open question. We first provide a characterization in a simple environment, before presenting the analysis in the general case.

3 Analytical results in the simple model

3.1 Model specification

To obtain a tractable framework we make three addiotnal assumptions. These ones are only valid in this analytical analysis and will be relaxed in the next Secion. The first assumption concerns the functional form of labor taxes that are assumed to be linear.

Assumption A *We assume that the labor tax is linear. Formally, we set in (2) $\tau_t = 0$ and denote $\tau_t^L := 1 - \kappa_t$, such that:*

$$T_t(\tilde{w}yl) := \tau_t^L \tilde{w}yl.$$

Our second assumption is about the utility function.

Assumption B *We assume that the instantaneous utility function U is of the GHH-type:*

$$U(c,l) := \log \left(c - \chi^{-1} \frac{l^{1+1/\varphi}}{1+1/\varphi} \right),$$

where $\varphi > 0$ is the Frisch elasticity of labor supply, and $\chi > 0$ scales labor disutility.

[4] A more general specification would consider the weights being a function of the whole history of idiosyncratic shocks for each agents: $\omega_t(y^{i,t})$. Such a generalization is not necessary in the quantatitve analysis, we hence follow the simpler formulation.

Assumption B, which is used in Diamond (1998) among others, simplifies the algebra for the Ramsey program by avoiding wealth effects for the labor supply. The log function also simplifies the derivatives.

The third assumption is about the productivity process.

Assumption C *We assume that there are only two productivity levels, equal to zero and one respectively:* $\mathcal{Y} = \{0, 1\}$*. Furthermore, the transition matrix is anti-diagonal:*

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{15}$$

while the initial distribution is such that: (i) a mass one of agents have productivity 1 with an identical beginning-of-period wealth; and (ii) a mass one of agents have productivity 0 with an identical beginning-of-period wealth (but possibly different from the one of employed agents).

The main implication of Assumption C is to simplify the equilibrium wealth distribution. First, there are only two productivity levels. The first one corresponds to a null productivity, and hence to a null labor supply. This zero productivity state will be called unemployment. The other productivity level is normalized to one and will correspond to employment. Second, equation (15) implies that the transitions to and out of unemployment are deterministic. Currently unemployed agents become employed in the next period and the other way around. Coupled with the assumption regarding the initial wealth distribution, Assumption C implies that at any date, the equilibrium only features two types of agents and two wealth levels. Our setup is thus similar to the one of Woodford (1990), in which 2 agents switch deterministically between employment and unemployment. For the sake of simplicity, the two types of agents will be called according to their current employment status: "employed" (subscript e) and unemployed (subscript u).

The fourth and last (not restrictive) assumption is about the credit constraint.

Assumption D *The credit-constraint is normalized to zero:* $\underline{a} = 0$*.*

Structure of the equilibrium

Taking advantage of Assumptions B–D, tractability is easy to derive. Using the peculiar equilibrium structure, the individual budget constraints (8) become:

$$c_{e,t} + a_{e,t} = R_t a_{u,t-1} + w_t l_{e,t}, \tag{16}$$

$$c_{u,t} + a_{u,t} = R_t a_{e,t-1}, \tag{17}$$

for employed (subscript e) and unemployed (subscript u), respectively. Note that the definitions (4) and (5) of the post-tax rates R_t and w_t are still valid (with $\tau_t = 0$ and $\kappa_t = 1 - \tau_t^L$). We can already state a first result regarding employed agents.

Result 1. *In any equilibrium, employed agents cannot be credit-constrained at any date.*

This is a direct consequence of budget constraint (17) with $c_{u,t} > 0$. Should we have $a_{e,t} = 0$ at some date t, we would have $c_{u,t+1} = -a_{u,t+1} \leq 0$, which would contradict the consumption strict positivity constraint. A consequence of Result 1. is that we only have two possible types of (steady-state) equilibria: one in which unemployed agents are not constrained, and one in which they are.

Taking advantage of the GHH property of the utility function and of the linearity of labor taxes, the labor Euler equation (11) for employed agents simplifies into:

$$l_{e,t} = (\chi w_t)^{\varphi}, \tag{18}$$

which only depends on the hourly wage w_t. The labor and financial market clearing conditions become in this set-up:

$$L_t = l_{e,t} \text{ and } B_t + K_t = a_{e,t} + a_{u,t}. \tag{19}$$

The governmental budget constraint (6) can be simplified using (18) and (19) as follows:

$$\begin{aligned} G_t + B_{t-1} + (R_t - 1)(a_{e,t-1} + a_{u,t-1}) + w_t(\chi w_t)^{\varphi} = \\ B_t + F(A_t, a_{e,t-1} + a_{u,t-1} - B_{t-1}, (\chi w_t)^{\varphi}). \end{aligned} \tag{20}$$

Finally, using budget constraints (16) and (17) and labor Euler equation (18), we deduce that Euler equations for consumption (10) can be expressed as:

$$\beta\left(R_t a_{u,t-1} - a_{e,t} + \frac{w_t(\chi w_t)^{\varphi}}{1+\varphi}\right) = a_{e,t} - a_{u,t+1}/R_{t+1}, \tag{21}$$

$$R_{t+1}a_{u,t} - a_{e,t+1} + \frac{w_{t+1}(\chi w_{t+1})^{\varphi}}{1+\varphi} \geq \beta R_{t+1}(R_t a_{e,t-1} - a_{u,t}), \tag{22}$$

where the first Euler equation holds with equality at all dates as a consequence of Result 1.. Expectations have been dropped from Euler equations due to MIT shocks and the deterministic transition between income levels.

We assume that the planner uses a utilitarian Social Welfare Function, maximizing the total welfare of the economy with the same wieght for all agents. The planner's aggregate welfare criterion can be expressed as:

$$W_0 = \sum_{t=0}^{\infty} \beta^t \left[\log\left(c_t^u\right) + \log\left(c_t^e - \chi^{-1}\frac{l_{e,t}^{1+1/\varphi} d}{1+1/\varphi}\right)\right], \tag{23}$$

where we have used the fact that there are only two consumption levels in equilibrium. The Ramsey problem is the choice of instruments $(\tau_t^K, \tau_t^L, B_t)_{t\geq 0}$ to select the competitive equilibrium maximizing the planner objective.

We now investigate the two possible types of Ramsey equilibria: one in which unemployed agents are not credit-constrained and which will correspond to the first best allocation, the other one in which unemployed agents are (optimally) constrained.

3.2 The first-best allocation

The first-best allocation is easy to derive. The planner maximizes (23) choosing c_t^e, c_t^u and $l_{e,t}$, with the resources constraint

$$c_t^e + c_t^u + G_t + K_t = K_{t-1} + K_{t-1}^{\alpha} l_{e,t}^{1-\alpha} - \delta K_{t-1} \tag{24}$$

Using standard techniques, one finds that the steady state first best allocation, provided in Appendix A.1. Allocation efficiency implies that the marginal utility of consumption is equalized between the two agents, the marginal disutility of labor is equal to its marginal poductivity, and the marginal productivity of capital is one over the discount facgor of the planner, $1/\beta$. One finds the first best level of production :

$$Y_{FB} \equiv \chi(1-\alpha)^{\varphi} \left(\frac{\alpha}{\frac{1}{\beta} + \delta - 1} \right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}$$

3.3 The case of low public spending : Decentralization of the first-best equilibrium

We first show that if the level of public spending is low enough, then the first best can be achieved. The intuition is simple enough. If G is low enough, the planner can hold a part of the capital stock and pay for the public spending with the interest perceived. In this case, it can set $\tau^K = \tau^L = 0$ and implement perfect consumption smoothing.

Define the value

$$\overline{g}_1 := \frac{1-\beta}{\beta} \frac{\alpha}{1/\beta + (\delta - 1)} - \frac{1-\beta}{1+\beta} \frac{1-\alpha}{\varphi + 1}, \tag{25}$$

then we have the Proposition :

Proposition 1 *If $G \leq \overline{g}_1 Y_{FB}$ the steady-stae Ramsey solution is the first-best steady-state equilibrium characterized zero taxes: $\tau^L = \tau^K = 0$, and perfect consumption smoothing.*

The proof can be found in Appendix A.2. The intuition for this thresold $\overline{g}_1$ is a condition on the total capital stock. In a production economy the planner could get some resources by holding the capital stock, facing the constraing that $-B \leq K$ (it cannot hold more than the actual capital sotck). This last constraint translates into a condition on parameters of the model.

For this reason, $\overline{g}_1$ is increasing in the capital share α, the Frish elasticity φ and decerasing in capital depreciation δ. The next Section focuses on the interesting case, where the first-best allocation cannot be implemented.

3.4 The equilibrium with binding credit constraint and positive capital taxation

We now turn to the only other equilibrium that admits a interior steady state.[5] To rule out the possibility of a first-best equilibrium, we make the following assumption.

Assumption E *We assume:*

$$G > \overline{g}_1 Y_{FB}.$$

We proceed by construction. This equilibrium features binding credit constraint for unemployed agents. In that case, unemployed agents hold no asset at any date: $a_{u,t} = 0$. The Euler equation (21) of employed agents implies:

$$a_{e,t} = \frac{\beta}{1+\beta} \frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} > 0, \tag{26}$$

which is positive whenever $w_t > 0$. Substituting the expression (26) of $a_{e,t}$ and using $a_{u,t} = 0$, the financial market clearing condition becomes:

$$B_t + K_t = \frac{\beta}{1+\beta} \frac{w_t(\chi w_t)^{\varphi}}{1+\varphi}. \tag{27}$$

With these expressions, one can now derive the optimal allocation. To simplify the Ramsey program of Section 2.6, we proceed in two steps. First, we use individual budget constraints (16) and (17) and the Euler labor equation (18) to express the Ramsey program in terms of savings choices and of the three instruments of fiscal policy $(w_t, R_t, B_t)_{t\geq 0}$. Second, we use the savings expression (26) of employed agents (that reflects employed agents' Euler equation) and $a_{u,t} = 0$ to express the Ramsey solely as a function of the fiscal policy $(B_t, w_t, R_t)_{t\geq 0}$ and with the government budget constraint, following the Chamley (1986). Indeed, consumption and labor supply can be written as a function of post-tax price only. After simple algebra and, we thus solve the equilibrium using post-tax price and then recover the pre-tax pices from the equilibrium

[5]We explain in Section A.7 below that no other equilibrium with interior steady state exists.

allocation :

$$\max_{(B_t,w_t,R_t)_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left((1+\beta)\log\left(\frac{1}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{\varphi+1}\right) + \log\left(\beta R_t\right)\right), \tag{28}$$

$$\text{s.t. } G + B_{t-1} + (R_t - 1)\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} + w_t(\chi w_t)^{\varphi} = \tag{29}$$

$$F\left(\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} - B_{t-1}, (\chi w_t)^{\varphi}\right) + B_t,$$

with furthermore the Euler inequality (22) stating than unemployed are actually credit-constrained. At the steady-state, this condition is equivalent to $\beta R < 1$ – which will always hold in this equilibrium. Two other constraints are implicit in the above program: (i) $w_t > 0$, and (ii) $R_t > 0$, which correspond to the positivity of consumption levels for employed and unemployed agents.

Before deriving first-order conditions, three important remarks are in order. First, this simple model allows for direct expression of the objective of the planner, but in more general models, this is not possible and the set of Euler equations must be used as a constraint. In this last case, one can use either the Lagrangian method used in LeGrand and Ragot (2022), or the primal approach used in Bhandari et al. (2020), if credit constrainsts don't occasionally bind. We show in Appendix B.1 that the first-order equations directly derived here, are the same as the ones derived from the Lagrangian method.

Second, even in this simple framework, one must show that the Karush–Kuhn–Tucker conditions apply to our problem, such that the first-order conditions charaterize an equilibrium (i.e. we have to check the constraint qualification) because the standard Slater (1950)'s conditions do not apply. This specific and standard difficulty stems from the non linearity of the constraint (29). This is done in Appendix A.3.

Third and in addition, one can check that the problem is concave, such that any interior optimum characterized by first-order conditions must be a maximum. This is done in Appendix A.4.

After these verifications, one can be confident that the first-order conditions of the above program charaterize the optimal solution of this problem.

The FOCs associated to the Ramsey program (28)–(29) can be written as, for $t \geq 0$ (See Appendix A.5 for the derivation):

$$1 = \mu_t \left(\frac{1}{1+\beta} - \varphi\frac{\tau_t^L}{1-\tau_t^L}\right) w_t \frac{(\chi w_t)^{\varphi}}{1+\varphi}, \tag{30}$$

$$\mu_t = \beta(1+\tilde{r}_{t+1})\mu_{t+1}, \tag{31}$$

$$1 = R_t \mu_t \frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi}, \tag{32}$$

where we still denote by $\beta^t \mu_t$ the Lagrange multiplier on the governmental budget constraint and also define w_{-1} as the solution of $a_{-1} = \frac{\beta}{1+\beta}\frac{w_{-1}(\chi w_{-1})^{\varphi}}{1+\varphi}$.

Equation (30) characterizes the labor tax, while (32) characterizes the capital tax. Equation (31) is an Euler-like equation for the Lagrange multiplier on the governmental budget constraint – and does not feature any expectation operator because of MIT shocks. Although obvious, it is useful to state that prices must be positive, such as the Lagrange multiplier on the budget of the stte μ_t for the equilibirum to exist (otherwise resources would be infinite for the planner).

Result 2. *The equilibrium with a binding credit constraint features the following restrictions:*

$$w_t, R_t, \mu_t > 0, \tag{33}$$

.

This obvious result is used in the proof of equilibrium existence below.

Steady-state analysis: Property We first characterize the steady state and then prove its existence. We will denote steady-state quantities with no subscript. For instance, R will be the steady-state gross post-tax interest rate. First, note that the restrictions of Result 3. still holds at the steady state. In particular, $\mu > 0$ implies from FOC (31) that:

$$\tilde{r} = \frac{1-\beta}{\beta}, \tag{34}$$

as in the first-best equilibrium. The previous equation is called the Modified Golden Rule since Aiyagari (1995), Proposition 1. This simple results comes from the fact the the planner uses public debt to smooth the marginal cost of tax distortions with the same discount factor as the agents, facing no financial frictions. Its Euler equation implies this efficient steady-state value. We also have, as in the first-best $K/L = K_{FB}/L_{FB}$, as well as $\tilde{w} = w_{FB}$, $w = (1-\tau^L)w_{FB}$, and $Y = (1-\tau^L)^{\varphi} Y_{FB}$. Using $R = 1 + (1-\tau^K)\tilde{r}$ and (34), we obtain the following expression for the capital tax:

$$\tau^K = \varphi \frac{1+\beta}{1-\beta} \frac{\tau^L}{1-\tau^L}, \tag{35}$$

which is an increasing function of the labor tax.

This equation is a key-equation of the simple model. Its states that in equilibrium (when it exists, see below), the labor and capital tax are related by a simple equation. There is a tradeoff for the planner between obtaining resources by raising labor tax, thus reducing labor supply and GDP, or raising resources by increasing capital tax, reducing the consumption smoothing across the two periods. Note that when the planneraffects the private saving, the public debt has to adjust to balance financial markets for the Modified Golden Rule (34) to be satisfied, and $K/L = K_{FB}/L_{FB}$. As a consequence, the labor tax mostly affects the *production efficiency* (the level of GDP), whereas the capital tax affects *allocation efficiency* (the consumption smoothing between periods). Relationship (35) shows that in equilibrium the capital tax increases with the labor tax : both distortions increase with the resources which have to be raised in the economy[6]. In particular, the capital tax is positive whenever the labor tax is.

The Straub-Werning threshold. The constraint, $R > 0$ (from result 3.) and the pre-tax interest rate (34) provide an upper bound for capital tax, denoted $\bar{\tau}^K_{SW} \equiv \frac{1}{1-\beta}$ (SW stands for Straub-Werning, see the discussion below) : $\tau^K < \bar{\tau}^K_{SW}$. Second the constraint $\mu > 0$ (also from result 3.) and the FOC (30) implies an upper bound for the labor tax, denoted as $\bar{\tau}^L_{SW} \equiv \frac{1}{1+(1+\beta)\varphi}$: $\tau^L < \bar{\tau}^L_{SW}$. These two thresholds are related by relationship (35). These two thresholds can be used to derive the maximum on the level of public spending. One finds that the constraint $G < \overline{g}_{SW} Y_{FB}$, with

$$\overline{g}_{SW} := \overline{g}_1 + (1-\alpha)\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}\right)(1-\bar{\tau}^K_{SW})^{\varphi}. \tag{36}$$

We summarize this in the following result, before the interpretation.

Result 3. *The equilibrium with a binding credit constraint exists only if $\tau^L < \bar{\tau}^L_{SW}$, $\tau^K < \bar{\tau}^K_{SW}$ and*

$$G < \overline{g}_{SW} Y_{FB}. \tag{37}$$

When the public spending is higher than the SW bound, the financing of public spending implies negative values of the steady Lagrange multiplier on government budget constraint. Such situations cannot be ruled out and are possible for some parametrizations. However, even though a steady-state equilibrium does not exist in these situations, we show that there exists a non-stationary equilibrium, where the Lagrange multiplier on the governmental budget constraint diverges to infinity: $\mu_t \to_t \infty$ and the gross interest rate converges to $R_t \to_t 0$ (See Appendix A.8). In other words, this situation is similar to the one in Straub and Werning (2020), where the stationary equilibrium doesn't exist but a non-stationary one does. It is noteworthy that a similar pattern emerges despite the differences between our set-ups. We have time-varying agents' types (although the switch deterministic), endogenous credit constraint, distorting tax on endogenous labor supply, and public debt. A key difference in our simple heterogeneous-agent model with Straub and Werning (2020) or with Lansing (1999) is that a steady-state equilibrium (including a finite Lagrange multiplier) exists for some public spending levels, even for log utilities and equal weight between agents. Proposition 2 shows that a crucial determinant for the existence of a steady-state equilibrium is the level of public spending.

The Laffer curve and the Laffer threshold Not all levels of public spending can be financed by distorting taxes,. At the steady state, in the equilibrium characterized by equations (30)-(32),

[6]One can check that τ^K/τ^L increases when the discount factor β and Frish elasticity of labor supply φ increases, what is intuitive.

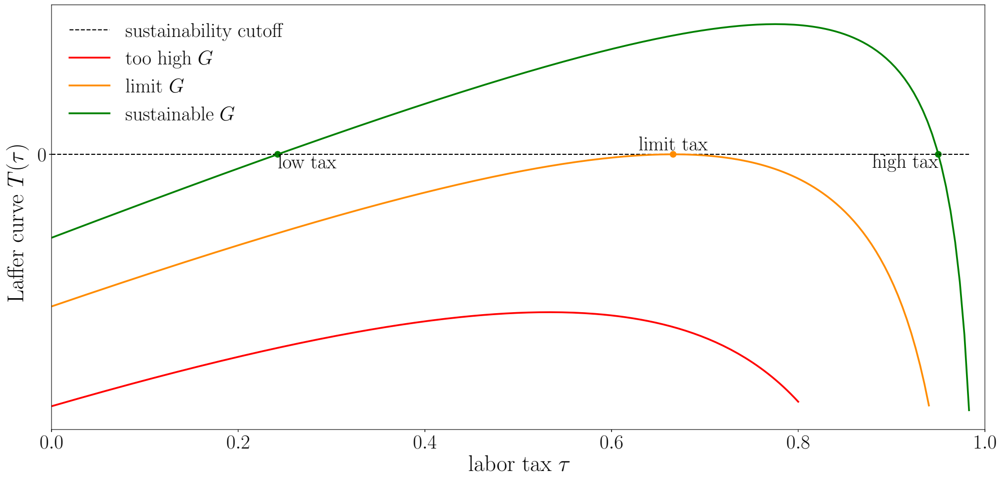

Figure 1: Examples of three Laffer curves for three different values of G. The three cases correspond to: (i) two admissible tax values; (ii) a unique limit tax value; (iii) no admissible tax. The parametrization is: $\beta = 0.97$, $\alpha = 0.3$, $\phi = 0.5$, $\delta = 1.0$, and G/Y_{FB} takes one of the three values in $[0.2, 0.3631, 0.6]$.

the planner budget constraint (29) implies, after some algebra, that the the labor tax τ^L is a solution of the following equation:

$$\tau^L - \frac{1}{1-\alpha}\frac{\frac{G}{Y_{FB}}(1-\tau^L)^{-\varphi} - \overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}} = 0$$

We study more formally this relationship[7] in Appendix A.6, where we show that this equation typically admits two roots, which correspond to the typical Laffer trade-off between tax rate and tax base. The smaller root corresponds to a low tax and a high labor supply, while the higher root corresponds to a high tax and a low labor supply. There is a limit value to public spending such that there is only one labor tax financing public spending. case that is the limit between sustainability and no sustainability. In this situation, there is a unique tax rate that enables public spending to be financed. Finally, if public spending is two high, then there is no solution. We plot the three possibilities in Figure 1.

As discussed in Appendix A.6, when there are two solutions for labor taxes, the planner chooses the lowest level to maximize welfare. We also show that the maximum normalized public spending G/Y_{FB} which can be financed is defined y the threshold $\overline{g}_{La}$:

$$\overline{g}_{La} := \frac{1-\alpha}{\varphi}\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}\right)\left(\frac{\varphi}{1+\varphi}\right)^{1+\varphi}(1 + \frac{1}{2}\frac{1+\varphi}{1-\alpha}\frac{\overline{g}_1}{\frac{1}{1+\beta}+\varphi})^{1+\varphi}, \tag{38}$$

We will henceforth refer to the restriction $G < \overline{g}_{La}Y_{FB}$ as the Laffer constraint. Note that

[7]As the Laffer curve is well known, we derive the Algebra in Appendix.

whether the Laffer constraint is more stringent than the SW constraint depends on parameters and in general both restrictions must be considered.

The steady-state equilibrium existence. We can provide our main result regarding steady-state equilibrium existence in the following Proposition.

Proposition 2 *When $\overline{g}_1 Y_{FB} \leq G$, $G \leq \overline{g}_{SW} Y_{FB}$, and $G < \overline{g}_{La} Y_{FB}$, there exists a steady-state equilibrium with binding credit constraint for unemployed agents where both taxes τ^L and τ^K are positive.*

We can verify that $\overline{g}_1 \leq \overline{g}_{SW}$ and $\overline{g}_1 \leq \overline{g}_{La}$. The former inequality is proved in Appendix , while the later is a direct implication of the definition (36). Therefore, the credit-constrained equilibrium always exists for some values of public spending. It can also be observed that when $G = \overline{g}_1 Y_{FB}$, equations (35) and (97) implies $\tau^L = \tau^K = 0$, as in the perfect risk-sharing equilibrium. As a consequence, there is no discontinuity between the first-best and the credit-constraint equilibria around $G = \overline{g}_1 Y_{FB}$.

3.5 The non-existence of the equilibrium with no-capital tax

Until now, we have only mentioned two equilibria : the first-best and the one where unemployed agents are credit-constrained and $\tau^K > 0$. We could thus wonder whether a full risk-sharing equilibrium with positive labor tax and hence null capital tax would exist. We can show that such an equilibrium doesn't exist because it is always dominated by an allocation where labor tax is reduced and capital tax is positive. The next proposition summarizes the result.

Proposition 3 *When the equilibirum with $\tau^K > 0$ exists (i.e. conditions of Proposition are fulfilled), then no steady-state equilibrium with $\tau^K = 0$ exists.*

The proof is in Appendix A.7, but the intuition can be simply provided. Imposing $\tau^K = 0$ implies full consumption-smoothing, but it also means that public spending should be solely financed bythe labor tax. The distortions implied by this high labor tax involves a high reduction in labor supply and private consumption. This makes the aggregate welfare lower than an equilibrium where public spending financing relies on both capital and labor taxes, where labor supply is higher but where consumption smoothing is not perfect. In other words, for any level of public spending, financing this public spending through a combination of capital and labor taxes generates smaller distortions than a financing relying solely on labor tax. The outcome can be seeen in the discussion of the relationship (35), which show that the pallner wants a positive capital tax when the labor tax is positive, balancing production and allocation efficiency.

3.6 When is optimal public debt positive ?

We now show that this simple model can explain both a positive capital tax and a positive amount of public debt. This result is not obvious: Why would the planner provide more public debt to the market (more liquidity in the sense of Woodford (1990)) and then tax the return on public debt with a positive capital tax ? The intuition is that capital tax is way to obtain some resources not increasing distorting labor tax. Hence, capital tax is a tool to improve production efficiency as discussed through equation (35). For this amount of capital and labor tax, public debt is a tool to absorbe extra saving when the saving of private agents is higher than the optimal level of capital stock. We now provide the algebra proving the claim that there exist equilibria with $\tau^L, \tau^K, B > 0$.

First, the financial market clearing condition implies that the steady-state public debt B can be written as follows, when credit constrained are biding for unempoyed agents:

$$B = \frac{\beta}{1-\beta}\left(-\frac{1-\beta}{1+\beta}\frac{1-\alpha}{1+\varphi}\tau_l - \overline{g}_1\right)(\chi w)^{\varphi}\left(\frac{K}{L}\right)^{\alpha}, \tag{39}$$

We deduce that the equilibrium features a positive public debt is positive iff:

$$\tau^L < \frac{1+\varphi}{1-\alpha}\frac{1+\beta}{1-\beta}(-\overline{g}_1). \tag{40}$$

Second, we can relate the fonstraint on τ^L to a constraint on public spending G. This is summarized in the following result.

Result 4. *Steady-state public is positive: $B \geq 0$ iff $\overline{g}_1 \leq 0$ and $G \leq \overline{g}_{pos} Y_{FB}$, where:*

$$\overline{g}_{pos} = (-\overline{g}_1)\frac{(1+2\varphi)(1+\beta)}{1-\beta}\left(\frac{(1+2\varphi)(1+\beta)}{1-\alpha}\frac{\alpha}{1+\beta(\delta-1)}\right)^{\varphi}. \tag{41}$$

The proof is in Appendix A.9. Several remarks are in order. First, the threshold $\overline{g}_{\text{pos}}$ depends positively on the capital share. In an economy without capital $\alpha = 0$, $\overline{g}_{\text{pos}} = 0$ and optimal public debt is always negative. It is only when the planner can get some resources out of capital income that optimal public debt can indeed be positive. Second, optimal public debt is positive only $\overline{g}_1 < 0$, which precludes from Proposition 1 the existence of the first-best equilibrium for the level of public spending G. Third, public debt is positive when public spending is not too high, and thus both labor and capital tax are not too high. Indeed, an increase in public spending leads the planner to increase labor and capital taxes, which are both distorting. This reduces private savings and hence diminishes the room for public debt, for the same targeted pre-tax marginal productivity of capital (34). Fourth and more intuitively, in an equilibrium with positive public debt, the equilibrium saving of employed agent is higher than the optimal capital stock, this extra saving being absorbed by public debt. From this allocation, decreasing public debt would inefficiently increase the capital stock, this would require and increase in capital tax to reduce saving, what would deteriorate consumption smoothing. [8]

3.7 Dynamic analysis of public debt in the relevant case

We now study the optimal dynamics of public debt after a public spending shock. To simplify the algebra, we focus on the case with full capital depreciation: $\delta = 1$. We denote with a hat the relative deviation to the steady-state value: $\widehat{x}_t = \frac{x_t - x}{x}$ for generic variable x_t with steady-state value x. The public spending shock is assumed to be defined as follows:

$$\widehat{G}_t = \begin{cases} \sigma_G \varepsilon_{G,0} & \text{if } t = 0, \\ \rho_G \widehat{G}_{t-1} & \text{if } t > 0, \end{cases} \tag{42}$$

$$\text{where: } \varepsilon_{G,0} \sim \mathcal{N}(0,1),$$

and $\sigma_G > 0$ and $\rho_G \in (-1, 1)$. The shock only happens at date $t = 0$ and then persists with parameter ρ_G – as is consistent with our assumption of MIT shock. Interestingly, we show in Appendix Appendix A.10.2 that the dynamic of the economy can be summarized by the capital as a unique state variable and the public spending shock. It can be computed thanks to a first-order development around the steady-state allocation. The outcome is gathered in the following result.

Result 5. *The dynamics of the capital stock is given by the following system:*

$$\widehat{K}_t = \rho_K \widehat{K}_{t-1} + \sigma_K \widehat{G}_t, \tag{43}$$

where $\rho_K > 0, \sigma_K < 0$, ρ_K does depend on ρ_G and $\frac{\partial \sigma_K}{\partial \rho_G} > 0$

The expressions of the coefficients are given in Appendix A.10.2. Thus at impact, an increase in public spending diminishes capital and the effect is higher, the higher the persistence of public spending shock.

The dynamic system (43) is stable when the autoregressive coefficient ρ_K is smaller than one in absolute value. In our setup, this is equivalent to verifying Blanchard-Kahn conditions. The result regarding system stability is summarized in the following proposition.

Proposition 4 *The system (43) is stable – $|\rho_K| < 1$ – iff:*

$$\alpha \leq \frac{1}{1 + (1 - \beta)(1 + \varphi)}. \tag{44}$$

The dynamic system is stable under the condition (44), which imposes an upper bound on α. Note that this upper bound is always strictly smaller than one and hence can be binding. This condition on α always holds when public debt is positive, i.e., when $\overline{g}_1 < 0$.

[8] To give an example where public debt is positive in this simple model ($G \leq \overline{g}_{\text{pos}} Y_{FB}$, $\overline{g}_1 Y_{FB} \leq G$, $G \leq \overline{g}_{SW} Y_{FB}$, and $G < \overline{g}_{La} Y_{FB}$), one can consider $\alpha = 0.2; \beta = 0.3; \varphi = 0.3; \delta = 1; G = 0.01; \chi = 1$.

By induction, we can derive from (42) and (43) the closed-form expression of the capital IRF:

$$\widehat{K_t} = \sigma_K \widehat{G}_0 \frac{\rho_K^{t+1} - \rho_G^{t+1}}{\rho_K - \rho_G}, \tag{45}$$

which allows us to completely characterize the capital path following a public spending shock. At impact, the relative variation of capital is negative by a quantity $\sigma_K \widehat{G}_0 < 0$. Then, the profile is the capital variation is humped-shaped: it starts decreasing further, before increasing and reverting back to zero. The length of the capital depreciation (during which $\widehat{K_t}$ diminishes following the initial shock) can be shown to be an increasing function of the persistence ρ_G: the higher ρ_G, the longer the recession. The impact on the depth of recessions is in general ambiguous, but it can be shown that (i) when the persistence is sufficiently high, it increases recession depth; (ii) the threshold value decreases with ρ_K. This is illustrated on Figure 2 below.

Role of the persistence of public spending shock ρ_G. We can now discuss the dynamics of public debt. As the algebra is involved, we first provide the intuition and then the Proposition.

First, the linearized dyamics of public debt is (from the financial market equilibrium $B_t = \frac{\beta}{1+\beta} \frac{\chi^\varphi}{1+\varphi} w_t^{1+\varphi} - K_t$) :

$$B\widehat{B}_t \;=\; \frac{\beta}{1+\beta}\chi^\varphi w^{1+\varphi}\widehat{w}_t - K\widehat{K_t}, \tag{46}$$

where B is the steady state level of public debt, which is assumed to be positive $B > 0$, to consider the relevant case. The dynamics of public debt is determined by difference between the post-tax real wage dynamics, $\widehat{w}_t$, which is affected by the labor tax τ_t^L and by the capital dynamics $\widehat{K_t}$. Assume that the planner implements income smoothing (to sustain consumption smoothing, as can be seen in the objective 28), in this case, the optimal fall in the capital stock would translate into an increase in public debt to absorb the excess saving, as can be seen in equation (46). When the persistence of the public spending shock increases, the planner deviates more and more from income smoothing to front-load the tax adjustment: At impact the higher the persistence of the shock, the higher the increase in the maginal value of public resources $\frac{\partial \widehat{\mu}_0}{\partial \rho_G} > 0$ (see inequality 132 in Appendix A.10). One can then show that the post-tax wage is falling $\frac{\partial \widehat{w}_0}{\partial \rho_G} < 0$ more the higher the persistence (equation 154). The capital stock also decreases on impact $\frac{\partial \widehat{K}_0}{\partial \rho_G} < 0$, but the front-loaded adjustment implies that the higher the persistence ρ_G, the higher the relative fall in the post-tax wage $\widehat{w}_0$ compared to the capital stock $\widehat{K}_0$. As a consequence, the higher the persistence, the lower the net increase in public debt $\widehat{B}_0$ on impact. The next proposition summarizes the result.

Proposition 5 *Denoting by $\widehat{B}_0$ the public debt variation on impact, we have:*

$$\frac{\partial \widehat{B}_0}{\partial \rho_G} < 0.$$

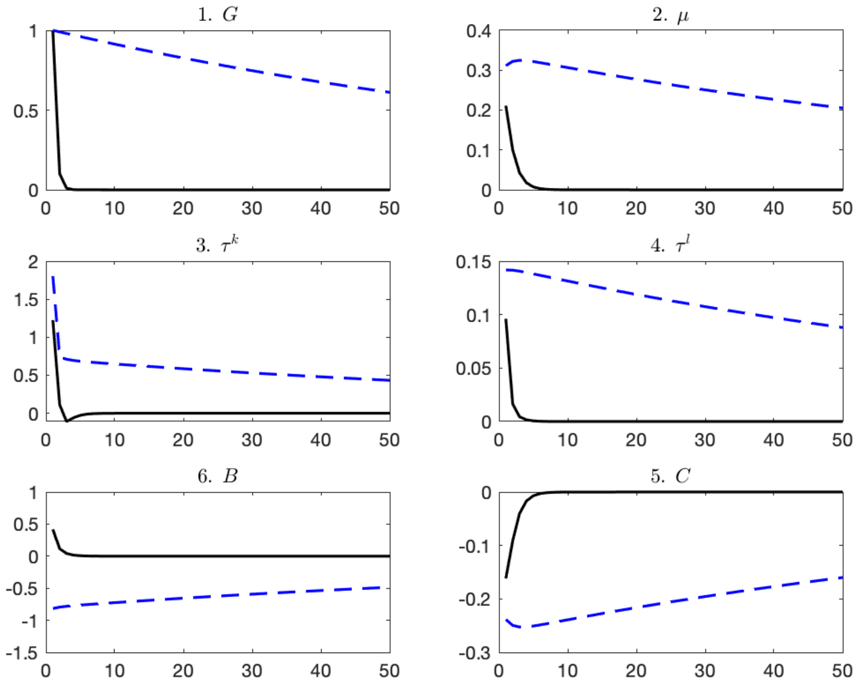

Figure 2: Examples of the dynamics of fiscal variables for two persistence $\rho_G = 0.1$ (black line) and $\rho_G = 0.99$ (blue dashed line). Parameters are $\alpha = 0.4, \beta = 0.97, \varphi = .5, \delta = 1, G = 0.05, \chi = 1$. Variables G, μ, C and B are in proportional change (in %) and τ^K, τ^L are in level change (in %).

The proof is in Appendix A.10.2, where the previous intuition is justified. A robust finding is that the public debt is decreasing with the persistence of the shock.

To save some space, we provide a suggestive example in Figure 2.

The figure plots the dynamics of the economy and of the instruments of the planner for the same economy, the same shock on impact, but different persistence. Parameters are $\alpha = 0.4, \beta = 0.97, \varphi = .5, \delta = 1, G = 0.05, \chi = 1$, one can check that $\bar{g}_1 Y_{FB} < G$, $G \leq \bar{g}_{SW} Y_{FB}$, and $G < \bar{g}_{La} Y_{FB}$. This economy has an equilibrium capital tax of 58%, labor tax of 14% and a slightly negative public debt of -0.01. The low-persistence economy is $\rho_G = 0.1$ and is the black solid line, the high-persistence economy is $\rho_G = 0.99$ and it is the blue dashed line. Panel 1 plots the increase in public spending. It increases by 1% on impact and then decreases back to its steady-state value. Panel 2. plots the increase in μ, the social value of public liquidity (i.e. the Lagrange multiplier on the government budget constraint). When the persistence is high, the increase in much high and persistent, compared to the low persistence case. Panel 3. plots the capital tax and Panel 4 the labor tax. When the persistence is high, both capital and labor taxes increases. Capital tax increases one order of magnitude more than the labor tax on impact, to front-load the adjustment as period 0 capital tax are not distorting (See Farhi (2010) for similar discussion). However, to avoid to reduce the resources of credit constrained agents, the planner doesn't fully front-load the adjustment and the labor tax is used on the whole transition. In the low-persistence economy, the change in the capital tax goes in negative territory after two periods (it decreases compared to the steady state value). Labor taxes are barely increasing in both economy (by less than 1%). As can be seen in Panel 5., public debt increases in the low-persistence economy, whereas it decreases in the high-persistence economy, as discussed in Proposition 5. Finally, panel 6. plots aggregate consumption. It falls in both case, but much more when the persistence is high.

From this example, we conclude that the dynamics of public debt depends on persistence of ths shock. Moreover, the size, and sometimes the sign of the change in taxes also depend on this persistence. We now consider a more general model to show that these results don't depend on the structure of this simple model.

4 The general model

We now solve the Ramsey allocation disposing of the simplifying assumptions of Section 3, to consider the general model of Section 2. More precisely, we now assume that i) the period utility function is separable in labor $U(c,l) = u(c) - v(l)$, what is empirically more relevant (see Auclert et al., 2021) ii) there are K idiosyncratic productivity level, and the transition matrix is a general Markov matrix, iii) the labor tax has a HSV structure $T_t(\tilde{w}yl) := \tilde{w}yl - \kappa_t(\tilde{w}yl)^{1-\tau_t}$, and iv) the Pareto wights may depend on productivity $\omega(y_t^i)$.

The Ramsey problem consists in choosing the fiscal instruments $(\tau_t^K, \kappa_t, \tau_t, B_t)_{t\geq 0}$ (as a function of the realization of the aggregate shock and of the initial distribution of the state variables of agents) which correspond to the competitive equilibrium with the highest aggregate welfare. Formally, the Ramsey program can be written as follows:

$$\max_{\left(r_t,w_t,B_t,K_t,L_t,(a_t^i,c_t^i,l_t^i,\nu_t^i)_i\right)_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \int_i \omega\left(y_t^i\right)(u(c_t^i)-v(l_t^i))\ell(di), \tag{47}$$

$$\tag{48}$$

$$G_t + R_t B_{t-1} + (R_t-1)K_{t-1} + w_t \int_i (y_t^i l_t^i)^{1-\tau_t}\ell(di) = F(K_{t-1},L_t) + B_t \tag{49}$$

$$\text{for all } i\in\mathcal{I}\text{: } a_t^i + c_t^i = R_t a_{t-1}^i + w_t (y_t^i l_t^i)^{1-\tau_t}, \tag{50}$$

$$a_t^i \geq -\bar{a},\ \nu_t^i(a_t^i+\bar{a}) = 0,\ \nu_t^i \geq 0, \tag{51}$$

$$U_c(c_t^i,l_t^i) = \beta\mathbb{E}_t\left[R_t U_c(c_{t+1}^i,l_{t+1}^i)\right] + \nu_t^i, \tag{52}$$

$$-U_l(c_t^i,l_t^i) = (1-\tau_t)w_t y_t^i (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i,l_t^i), \tag{53}$$

$$K_t + B_t = \int_i a_t^i \ell(di),\ L_t = \int_i y_t^i l_t^i \ell\left(di\right), \tag{54}$$

The Ramsey program consists for the planner to maximize aggregate welfare W_0 subject to the governmental budget constraint (48) and to the constraints characterizing the competitive equilibrium: individual budget constraints (8), individual Euler equations (10) and (11), individual credit and positivity constraints (9), market clearing conditions (12) and factor price definitions (1), (4), and (5). We solve this program using a Lagrangian approach, presented in LeGrand and Ragot (2022).[9]

We denote as $\beta^t\lambda_{c,t}^i$ the Lagrange multiplier on the period t Euler equation of agents i, equation (52). When the credit constraints of agents i is binding $a_t^i = -\bar{a}$, and $\lambda_{c,t}^i = 0$, as the Euler equation is not a constraint. It is shown in LeGrand and Ragot (2022) that (when the credit constraint does not bind), the equilibrium can feature either $\lambda_{c,t}^i > 0$ or $\lambda_{c,t}^i < 0$ depending on whether the agents save too much or too little *seen from* the planner perspective. Similarly, we denote by $\beta^t\lambda_{l,t}^i$, the Lagrange multiplier on the labor supply (53), and by $\beta^t\mu_t$ the Lagrange multiplier on the government budget constraint (48)

To save some place, we derive the first-order conditions of the planner in Appendix B. Note that we follow the literature and assume the solution are interiors and first-order conditions of the planner are sufficient to characterize the optimal allocation. We provide some quantitative checks below.

To simplify the interpretation of the first-order conditions of the Ramsey program, we

[9] In LeGrand and Ragot (2022), we show that this method can be used with occasionally binding credit constraints, taking limits of penalty functions. See also Açikgöz et al. (2018) to solve for policies with a utilitarian social welfare function.

introduce the marginal social valuation of liquidity for agent i, defined as:

$$\psi_t^i := \omega_t^i U_c(c_t^i, l_t^i) - \left(\lambda_{c,t}^i - (1+r_t)\lambda_{c,t-1}^i\right) U_{cc}(c_t^i, l_t^i) \\ + \lambda_{l,t}^i \left(U_{cl}(c_t^i, l_t^i) - (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t} U_{cc}(c_t^i, l_t^i)\right). \tag{55}$$

This complex expression has a simple interpretation. It is the net value for the planner of transferring one unit of resources to agents i (if it could). First, the gain for the planner would be to increase marginal utility, weighted with the relevant weight $\omega_t^i U_c(c_t^i, l_t^i)$. Second, one additional unit of resources to agent i changes the incentive to save from period $t-1$ to period t, captured by the term with $\lambda_{c,t-1}^i$. Third, it also affects the incentive to save from period t to period $t+1$, captured by the term with $\lambda_{c,t}^i$. Fourth, it affects the incentive to work , captured by the terms in $\lambda_{l,t}^i$. For these last three terms, the effect is multiplied by the marginal change in the marginal utility of consumption, which is the term $U_{cc}(c_t^i, l_t^i)$.

From (55), we also define the net social valuation of liquidity that accounts for the opportunity cost of liquidity, measured by the Lagrange multiplier :

$$\hat{\psi}_t^i := \psi_t^i - \mu_t. \tag{56}$$

With this notation, the first order conditions of the planner can be easily interpreted. First, for an unconstrained agent i, the planner implements a liquidity smoothing condition:

$$\hat{\psi}_t^i = \beta \mathbb{E}_t R_{t+1} \hat{\psi}_{t+1}^i, \tag{57}$$

where the expectation is taken with respect to the idiosyncratic risk. Equation (57) is a generalized version of the Euler equation (10) (and it is actually the same equation, when all Lagrange multipliers are 0), in which the planner internalizes in the definition of $\hat{\psi}_t^i$ the general equilibrium externalities when setting individual savings.

The first-order condition with respect to labor can be written as:

$$\psi_{l,t}^i = (1-\tau_t) w_t y_t^i (y_t^i l_t^i)^{-\tau_t} \hat{\psi}_t^i \\ + \mu_t F_{L,t} y_t^i - \lambda_{l,t}^i (1-\tau_t) \tau_t w_t y_t^i (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i)/l_t^i, \tag{58}$$

where we have defined:

$$\psi_{l,t}^i := -\omega_t^i U_l(c_t^i, l_t^i) - \lambda_{l,t}^i U_{ll}(c_t^i, l_t^i) \\ + (\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i - \lambda_{l,t}^i (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t}) U_{cl}(c_t^i, l_t^i). \tag{59}$$

Similarly to ψ_t^i for consumption, the quantity $\psi_{l,t}^i$ is the social marginal value of labor supply by agent i. The Ramsey first-order condition (58) is a generalized version of the labor Euler equation (11).

The first-order condition with respect to public debt can be written as:

$$\mu_t = \beta(1 + \tilde{r}_{t+1})\mu_{t+1}, \tag{60}$$

without expectation operator thanks to the MIT shock assumption. Equation (60) shows that the planner aims at smoothing the shadow cost of the government budget constraint through time.

The other first-order conditions with respect to R_t, w_t, and τ_t can respectively be written as:

$$0 = \int_j \left(\hat{\psi}_t^j a_{t-1}^j + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j)\right) \ell(dj), \tag{61}$$

$$0 = \int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j\right) \ell(dj), \tag{62}$$

$$\begin{aligned} 0 = &\int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j\right) \ln(y_t^j l_t^j)(dj) \\ &+ \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t} (U_c(c_t^j, l_t^j)/l_t^j) \ell(dj). \end{aligned} \tag{63}$$

All these equations have a similar interpretation. They involve equalizing the net valuation of liquidity weighted aggregated over the whole population with the relevant weight (e.g., $\int_j \hat{\psi}_t^j a_{t-1}^j \ell(dj)$ in the case of the interest rate) to the general-equilibrium distortion of the instrument (e.g., distortion of savings incentives for the interest rate).

In Appendix B.1, we check that the application of this Lagrangian approach to the environment of Section 3 deliver the same FOCs (30)-(32). Considering the general requires the above generalization.

The analytical characterization of the dynamics is a first step to determine the optimal policy. However, standard recursive techniques cannot be used to compute the policy in the general model. The problem of the planner could be written recursively, but in this case the state space would include the joint distribution of beginning-of-period wealth and Lagrange multipliers on consumption Euler equations (i.e., the joint distribution of $(a_{t-1}^i, \lambda_{c,t-1}^i)_i$). Indeed, beginning-of-period wealth a_{t-1}^i and past value of the Lagrange multiplier λ_{t-1}^i both appear in the first-order conditions of the Ramsey program. To compute the solution, we again follow LeGrand and Ragot (2022) and we consider a truncated representation of this problem. We provide the details of of the analytical implementation of the truncation approach in Appendix C. In Le Grand and Ragot (2022), we prove the acuracy of the truncation approach to solve for optimal policies, we thus skip this case here.

5 Quantitative analysis of the general model

We now show that the intuitions concerning the dynamics of public debt, derived in the simple environment are valid considering a realistic calibration of the general setup. The quantitative strategy is as follows. First to calibrate standard parameters to obtain a realistic steady-state allocation considering parameters of the actual US fiscal policy. Second, following the inverse taxation problem (Bourguignon and Amadeo, 2015, Heathcote and Tsujiyama, 2021, Chang et al., 2018), we estimate an "empirically motivated" social welfare function, such that this steady-state allocation is optimal for the planner. The gain of this methodology is to observe the dynamics of the tax system, considering a quantitatively realistic initial allocation. Starting from this allocation, we implement a period-0 shock on public spending to observe the dynamics of fiscal instruments after the public spending shock.

5.1 Calibration

The period is a quarter.

Preferences. The utility function is separable in labor $U(c,l) = u(c) - v(l)$, where

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \text{ and } v(l) = \frac{1}{\chi}\frac{l^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}}.$$

We set the inverse of intertemporal elasticity of substitution to $\sigma = 2$, which is a standard value used in the literature. For the disutility of labor, we choose $\phi = 0.5$ to match a Frisch elasticity for labor supply of 0.5, which is the value recommended by Chetty et al. (2011) for the intensive margin in heterogeneous-agent models. The scaling parameter is set to $\chi = 0.05$, which implies normalizing the aggregate labor supply to 1/3. Finally, the discount factor is $\beta = 0.99$.

Idiosyncratic risk. We focus on a standard AR(1) process:

$$\log y_t = \rho_y \log y_{t-1} + \varepsilon_t^y,$$
$$\text{where: } \varepsilon_t^y \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_y^2).$$

This process is discretized in 7 idiosyncratic states. Following the strategy of Castaeneda et al. (2003), we choose the parameters (ρ_y, σ_y) to target some key moments.[10] As Castaeneda et al. (2003), we calibrate the labor process to match three relevant targets. The first one is the variance of the logarithm of consumption, that enables us to capture consumption inequality. Heathcote

[10] More precisely, we minimize the quadratic difference between the model-generated moments and their empirical counterpart, following the Simulated Method of Moments. In the current environment, we see this procedure as a "sophisticated" calibration, rather than an actual SMM – as we equally weight the three moments.

and Tsujiyama (2021) report a value of $\text{Var}(\log c) = 0.23$. We also target the log-variance of wages to match income inequality, which is found to be $\text{Var}(\log w) = 0.47$ by Heathcote and Tsujiyama (2021). The third target is the debt-to-GDP ratio which allows us to replicate a realistic financial market equilibrium. We target a value of $B/Y = 61.5\%$, which is the mean ratio over the period (Dyrda and Pedroni, 2018). Calibrating these three moments yields $\rho_y = 0.993$ and $\sigma_y = 0.082$. These parameters are close to those from a direct estimation of the productivity process on PSID data, which corresponds to $\rho_y = 0.9923$ and $\sigma_y = 0.0983$ (see Boppart et al., 2018 and Krueger et al., 2018). The data targets and their model counterparts are reported in Table 1. This simple representation is doing a good job in matching the three targeted moments.

	Data	Model
Variance of log consumption Var(log$c$)	0.23	0.20
Variance of log income Var(log$y$)	0.47	0.49
Debt-to-GDP ratio B/Y	61.5%	61.4%

Table 1: Model calibration: targets and model counterparts.

Furthermore, we can check that this calibration generates a reasonable wealth distribution, even though we do not calibrate it explicitly.[11] Indeed, the calibrated model implies a Gini coefficient of wealth equal to 0.66, which is close, even though below, its empirical counterpart of 0.77. It is known that additional model features must be introduced to match the high wealth inequality in the US, such as heterogeneous discount rates (see Krusell and Smith, 1998), or entrepreneurship (Quadrini, 1999), or stochastic financial returns, which are not considered here.

Finally, we discretize the productivity process using the Rouwenhorst (1995) procedure with 7 idiosyncratic states.

Technology . The production function is Cobb-Douglas: $F(K,L) = K^{\alpha}L^{1-\alpha} - \delta K$. The capital share is set to $\alpha = 36\%$ and the depreciation rate to $\delta = 2.5\%$, as in Krueger et al. (2018) among others.

Taxes and government budget constraint. The capital tax is taken from Trabandt and Uhlig (2011), who use the methodology of Mendoza et al. (1994) on public finance data prior to 2008. Their estimation for the US in 2007 (before the financial crisis) yields a capital tax (including both personal and corporate taxes) of $\tau^K = 36\%$. For the labor we consider the HSV functional form of equation (2). The progressivity of the labor tax is taken from Heathcote et al.

[11] For the problem under consideration, we consider matching the dispersion of consumption may be more important than the distribution of wealth, which motivates the exclusion of this moment from our calibration strategy.

(2017), who report an estimate $\tau = 0.181$. We choose κ to match a public-spending-to-GDP ratio equal to 19%, as in Heathcote and Tsujiyama (2021).

Summary. Table 2 provides a summary of the model parameters.

Parameter	Description	Value
	Preference and technology	
β	Discount factor	0.99
α	Capital share	0.36
δ	Depreciation rate	0.025
$\bar{a}$	Credit limit	0
χ	Scaling param. labor supply	0.05
φ	Frisch elasticity labor supply	0.5
	Shock process	
ρ_y	Autocorrelation idio. income	0.993
σ_y	Standard dev. idio. income	0.082
	Tax system	
τ^K	Capital tax	36%
κ	Sacaling of Labor tax	0.75
τ	Progressivity of tax	0.181

Table 2: Parameter values in the baseline calibration. See text for descriptions and targets.

5.2 Truncation and estimating Pareto weights

We provide a detailed account of the computational implementation of the truncation method in Appendix C, which can be of independent interest as solving such Ramsey problems is not straightforward. More precisely, to investigate the optimal dynamics of the instruments after a shock, we start with providing an exact truncated aggregation of the steady-state model, and we then follow the dynamics of the truncated representation using perturbation methods.

The truncation length is set to $N = 3$, which is shown to be provide a good representation of the dynamics. We thus consider $7^3 = 343$different histories. We have to estimate the weights of the social welfare function, such that the first-order conditions of the planner at the steady state are consistent with actual US tax system (as described in Section 5.1). However, the problem is in general under-identified, since we have only two constraints (for the capital and labor tax) but seven different weights (one per productivity level). Following Heathcote and Tsujiyama (2021), we introduce productivity weights which depend on the productivity level and define a parametric quadratic representation of weights, as follows:

$$\log \omega_y := \theta_1 \log y + \theta_2 (\log y)^2 .$$

As explained in Appendix C, matching capital and labor tax yields $\theta_1 = 0.95$ and $\theta_2 = 0.66$. In an environment without saving, Heathcote and Tsujiyama (2021) estimate the relationship $\log \omega_y = \theta \log y$ and find a positive value $\theta = 0.517$. The quantitative difference mostly comes from the additional instruments we use.[12]

5.3 Model dynamics

We now simulate the optimal dynamics of the four fiscal tools $(\tau_t^K, B_t, \kappa_t, \tau_t)$ after a public spending shock occurring in period $t = 0$. The dynamics of the shock is the same as in equation (42) of the analytical section. After a initial shock in period 0, public spending reverts back to equilibrium at a rate ρ_G.

We first plot the dynamics of the model for two values of the persistence of the shock. The high one is $\rho_G = 0.97$. This value corresponds to the annual value used by Farhi (2010), which is an estimate of US data. The low value is $\rho_G = 0.6$, which correspond to some very specific very transitory increase in public spending in the US, as specific episodes of military build-ups. Figure 3 first plots the dynamics of shocks and the instruments. It plots public spending shock G, the Lagrange multiplier μ, both in proportional deviations, the labor level, κ, and the progressivity parameter τ, the capital tax, both in level deviations and finally public debt B in proportional deviations. The high-persistence value is in dashed blue. The low-persistence value is in black solid line.

Panel 1 represents the dynamics of public spending in proportional deviation. It increases by 1% and goes back to equilibrium at a rate $\rho_G = 0.6$ (black solid line) or $\rho_G = 0.97$ (blue dashed line). Panel 2 plots the value of the Lagrange multiplier (in proportional deviation), which represents the marginal value of additional public resources. The increases in μ four times more when the persistence is high (for the same impact increase in G), as it captures the intertemporal value of the increase in public spending. Panel 3, 4 and 5 report labor tax, tax progressivity and capital tax (in level deviation). All of these variables increase when the persistence is high, but they decrease when persistence is low (what is a difference with the simple model). When the persistent is high, the planner increases both taxes and progressivity on impact to levy some resources and decrease public debt (Panel 6.). When the persistent is low, the planner decreases taxes and progressitivity and public debt increase to finance public spending. The difference concering labor tax between the simple amodel (Figure 2) and this general model comes from the assumption of GHH utility function in the simple model, which is known to bias the fiscal

[12] We cannot strictly reproduce the specification of Heathcote and Tsujiyama (2021) within our framework, as we need two parameters to match planner's first-order conditions, because we have more instruments. The correlation between the estimated value $\log \omega_y$ and $\log y$ is 0.68 in our model, close to Heathcote and Tsujiyama (2021).

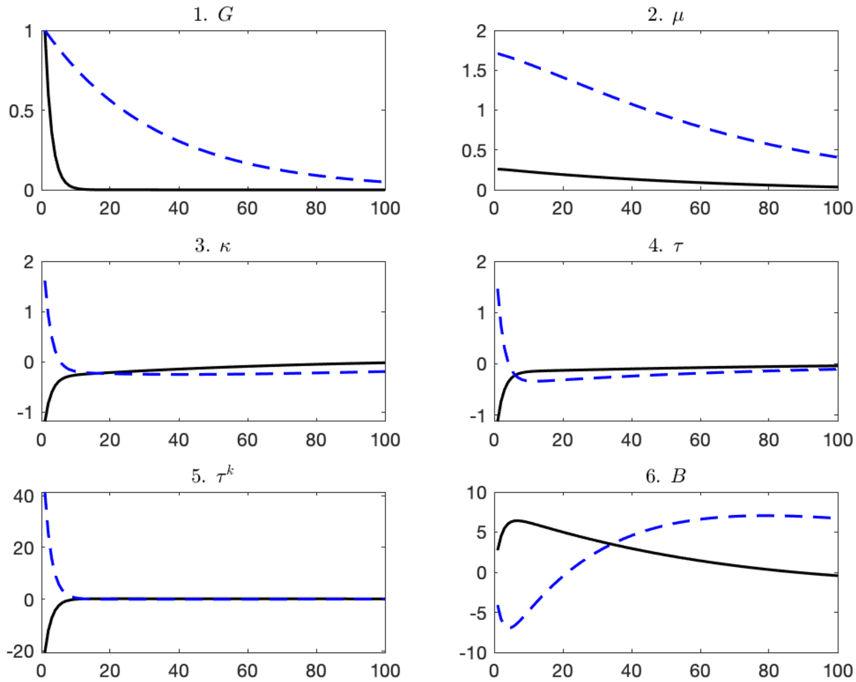

Figure 3: Dynamics of selected variables. Public spending G, value of public resources μ, the level of labor tax κ, progressivity of labor tax τ, capital tax τ^k and public debt B. The black solid line is for the persistence $\rho_G = 0.6$. The blue dashed line is for persistence $\rho_G = 0.97$. G, B are in proportional deviations, other variables are in proportional deviations.

system in favor of labor taxes for public spending shocks. Public debt is represented in Panel 6 in proportional deviation. When persistence is low, public debt actually increases to 6% after four quarters, and then it goes smoothly back to its steady-state value. When persistence is high, public debt decreases on impact. It first decreases for roughly 20 quarters (by 7% at minimum), and then it increases to 7% before converging back to its long-run value after long period of time, roughly 400 quarters. This non-linear dynamics comes from the noticeable tax-smoothing outcome, even in the case of high persistence. When the persistence is high, the autocorrelation of τ, τ^k, and κ are lower than 0.85, much lower than the persistence of G (0.97), whereas the autocorrelation of public debt is high (0.998).

It is noticeable that in both cases (hogh and low persistence) the planner implements a significant change in taes for few quarters, and then let taxes and progressivity converge rapidly to their equilibrium value. Public debt exhibits much more persistent deviations.

Figure 4 plots the dynamics of aggregate variables for the the two same economies (blue dashed line is the high persistence case, the black solid line is the low-persistence case, both represented in Panel 1. of Fig 3). It plots output Y, capital K, labor $L$and total private consumption C, all in proportional deviation.

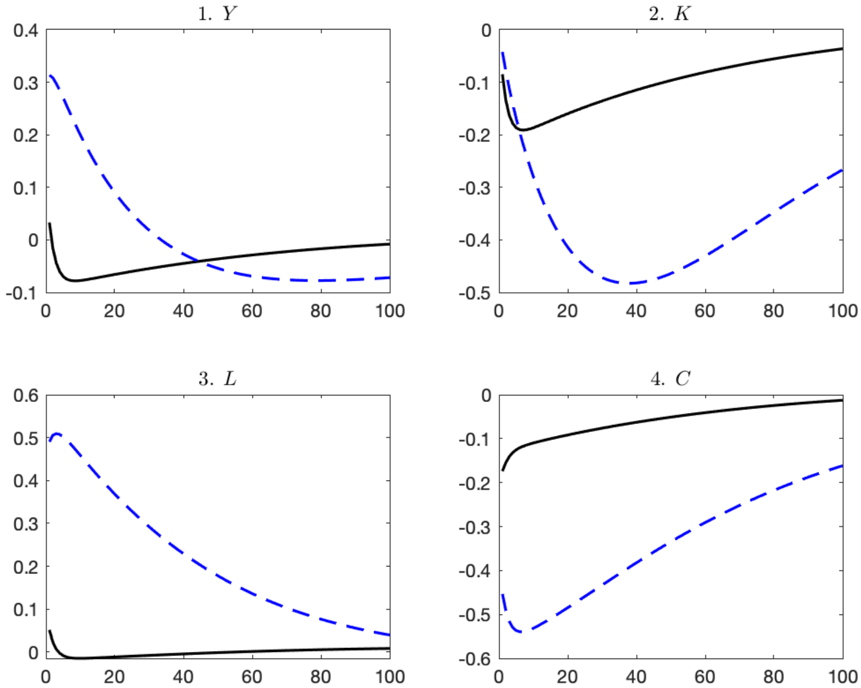

Figure 4: Output, Y, capital K, labour L and consumption. The blue dashed line is for persistence $\rho_G = 0.97$. All variables are in proportional deviations.

First output increases more when the persistence is high. The impact multiplier $\partial Y/\partial G$ is then around 0.3, a value lower than 1, due to the fall in consumption[13], which can be seen in Panel 4. The capital stock falls by 0.37 after 30 quarters. Total labor supply is increasing (due to a negative wealth effect for households) increases by roughly 0.52%. When the persistence is low, output increases just a little bit on impact (hence the multiplier is lower). For other variables the effects are mutated, with the same direction.

To conclude, the dynamic of the fiscal system in heterogeneous-agent models depends on the persistence of the shock, and not only the initial increases of public spending nor its total value. In the case of high persistence, the planner front load the adjustment, and hence increase taxes on impact and implement a decrease in public debt in the first quarters. In the case of low persistence, a decrease in taxes is implement for consumption smoothing and public debt increases temporarily. Finally, although the simple model of Section 3 provides relevant intuitions, some qualitative aspects of tax changes (such as the evolution of labor taxes) require a more general model and a realistic utility function.

[13]We also computed the cumulative multiplier, which are also an increasing function of persistence.

6 Conclusion

We investigate the optimal dynamics of fiscal system after a public spending shock in an heterogeneous agent model. We first contribute to the clarification of the conditions for relevant equilibria to exist. The key friction for equilibrium existence is occasionally-binding credit constraint, which provide a rationale for both positive capital tax and public debt. The second contribution of this paper is to show that the dynamics of public debt and taxes depends crucially on the persistence of the public spending shock. For low persistence, public debt is pro-cyclical. For high persistence; the public debt is countercyclical. In the general model, we find that both capital and labor tax increases when persistence is high, and decreases otherwise. We consider a quantitative model where the actual US tax system is implemented at the steady state thanks to an inverse optimal taxation approach. The simulation of the quantitative model relies on the Lagrangian-Truncation approach developed in LeGrand and Ragot (2022).

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Appendix

A Property of the simple model

A.1 First-best steady-state allocation in the simple model

We derive the first-best allocation of the simple model. Considering the Utilitarian Socia Welfare Function, the Lagrangian associated to the program is simply:

$$\begin{aligned}\mathcal{L}^{FB} &= \sum_{t=0}^{\infty}\beta^t\left[\log\left(c_t^u\right)+\log\left(c_t^e-\chi^{-1}\frac{l_{e,t}^{1+1/\varphi}}{1+1/\varphi}\right)\right]\\ &+ \sum_{t=1}^{\infty}\beta^t\mu_t\left(K_{t-1}+K_{t-1}^{\alpha}l_{e,t}^{1-\alpha}-\delta K_{t-1}-c_t^e-c_t^u-G_t-K_t\right),\end{aligned}$$

together with non-negativity constraints $c_t^e, c_t^u, l_{e,t} \geq 0$, which are not binding.

To ease the interpretation, we call $L_{FB,t} = l_{e,t}$ the first-best labor supply in this economy. Deriving the FOCs yield, after simple manipulations

$$\frac{1}{c_t^u} = \frac{1}{c_t^e-\chi^{-1}\frac{l_{e,t}^{1+1/\varphi}}{1+1/\varphi}} = \mu_t \tag{64}$$

$$l_{e,t}^{1/\varphi} = \chi\left(1-\alpha\right)K_{t-1}^{\alpha}l_{e,t}^{-\alpha} \tag{65}$$

$$\mu_t = \beta\left(\alpha K_{t-1}^{\alpha-1}l_{e,t}^{1-\alpha}+1-\delta\right)\mu_{t+1} \tag{66}$$

At the steady state we have the well-known equations:

$$\begin{aligned}\frac{K_{FB}}{L_{FB}} &= \left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{1}{1-\alpha}}\\ L_{FB} &= \left(\chi\left(1-\alpha\right)\right)^{\varphi}\left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{\alpha}{1-\alpha}\varphi}\\ Y_{FB} &= \chi(1-\alpha))^{\varphi}\left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}\end{aligned}$$

Then using the fact that $c_t^u = c^e - \chi^{-1}\frac{L_{FB}^{1+1/\varphi}d}{1+1/\varphi}$ (from 64) and the resource constraint (24), one can derive consumption shares.

A.2 Proof of Proposition 1

The first-best equilibrium is characterized by optimal consumption smoothing and no inefficient distortions. We now analyse the necessary and sufficients allocation for which the first-best allocation can be decentralized. Using the Euler equations (21) and (22) and consumption smoothing $u'\left(c_e - \frac{1}{\chi}\frac{l_e^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right) = u'\left(c_e - \frac{1}{\chi}\frac{l_e^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right)$, one finds

$$\beta R_{FB} = 1, \tag{67}$$

To get prodcution and allocation efficiency, distorting taxes must be null $\tau^K = \tau^L = 0$, while the government budget constraint (20) implies that the public debt verifies:

$$B_{FB} = -\frac{\beta}{1-\beta}G < 0.$$

The previous condition is necessary but not sufficient to ensure that the first-best allocation can be implemented. Indeed, the additional constraint is that no agents are strictly constrained. We now derive this additional condition.

Factor prices definitions (1) with (67) and $L_{FB} = l_e = (\chi w_{FB})^{\varphi}$ yield:

$$\frac{K_{FB}}{L_{FB}} = \left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{1}{1-\alpha}}, \tag{68}$$

from which we easily deduce:

$$w_{FB} = (1-\alpha)\left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{\alpha}{1-\alpha}}, \tag{69}$$

$$Y_{FB} = K_{FB}^{\alpha}L_{FB}^{1-\alpha} = (\chi(1-\alpha))^{\varphi}\left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}, \tag{70}$$

$$K_{FB} = \left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{1}{1-\alpha}}(\chi w_{FB})^{\varphi}. \tag{71}$$

Furthermore, since agents are unconstrained, Euler equations imply $c_{u,FB} = c_{e,FB} - \frac{1}{\chi}\frac{l_{e,FB}^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}$, or after substituting by budget constraints: $R_{FB}a_{u,FB} - a_{e,FB} + \frac{w(\chi w)^{\varphi}}{\varphi+1} = R_{FB}a_{e,FB} - a_{u,FB}$. With (67), this yields:

$$a_{e,FB} - a_{u,FB} = \frac{\beta}{1+\beta}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1}, \tag{72}$$

$$a_{u,FB} + a_{e,FB} = K_{FB} - \frac{\beta}{1-\beta}G, \tag{73}$$

where the second equality is the financial market clearing condition. The combination of both previous equations implies:

$$2\frac{1-\beta}{\beta}\frac{a_{u,FB}}{Y_{FB}} = \overline{g}_1 - \frac{G}{Y_{FB}}, \tag{74}$$

$$\text{with: } \overline{g}_1 = \frac{1-\beta}{\beta}\frac{\alpha}{1/\beta + (\delta - 1)} - \frac{1-\beta}{1+\beta}\frac{1-\alpha}{\varphi+1}, \tag{75}$$

Due to the the credit constraint $a_{u,FB} \geq 0$, if the first-best equilibrium exists, equation (74) implies that $\frac{G}{Y_{FB}} \leq \overline{g}_1$. We can then deduce $a_{e,FB}$ from (72):

$$a_{e,FB} = a_{u,FB} + \frac{\beta}{1+\beta}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1},$$

which verifies $a_{e,FB} \geq a_{u,FB} \geq 0$.

A.3 Constraint qualification

In our problem, even though the objective function is concave, the equality constraints are not linear and the standard Slater (1950)'s conditions do not apply. However, we can check that the linear independence constraint qualification (LICQ) holds in our problem. This constraint qualification requires the gradients of equality constraints to be linearly independent at the optimum (or equivalently that the gradient is locally surjective). At any date t, two constraints matter for the instruments of date t. These are the constraints at dates t and $t+1$. We can check that their gradient can be written as:

$$\begin{pmatrix} 1 & \varphi(\chi w_t)^{\varphi}\frac{\tilde{w}_t}{w_t} - (\varphi+1)(\chi w_t)^{\varphi} & -\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} \\ -\tilde{r}_{t+1} - 1 & \frac{\beta}{1+\beta}(\chi w_t)^{\varphi}\tilde{r}_{t+1} - (R_{t+1}-1)\frac{\beta}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} & 0 \end{pmatrix}, \tag{76}$$

which forms a matrix of rank 2. Indeed, looking at first and third columns of the matrix in (76) makes it clear that a sufficient condition is $(1+\tilde{r}_{t+1})w_{t-1} \neq 0$. This condition must hold at the optimum, since: (i) equation (1) implies $\tilde{r}_{t+1} \geq 0$, and (ii) we must have $w_{t-1} > 0$.

A.4 Second-order conditions

In the program (28)–(29), we can use the constraint (29) to substitute for the expression of R_t. We can further use financial market constraint (27) to express the public debt B_t as a function of capital K_t and wage post-tax wage w_t. The planner's program (82)–(83) can be equivalently rewritten as a function of K_t and w_t:

$$\max_{(K_t,w_t)_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log\left(w_t(\chi w_t)^{\varphi}\right) + \log(K_{t-1} + F(K_{t-1}, (\chi w_t)^{\varphi}) \right.$$
$$\left. + \frac{\beta}{1+\beta} \frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} - K_t - G_t - w_t(\chi w_t)^{\varphi}) \right).$$

We can further modify this program by defining $W_t = w_t(\chi w_t)^{\varphi}$ and dropping constants:

$$\max_{(K_t,w_t)_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg(\log(W_t) \tag{77}$$
$$+ \log\left(K_{t-1} + F(K_{t-1}, \chi^{\frac{\varphi}{1+\varphi}} W_t^{\frac{\varphi}{1+\varphi}}) - \frac{1+\varphi+\varphi\beta}{(1+\beta)(1+\varphi)} W_t - K_t - G_t \right) \Bigg). \tag{78}$$

The function $(W_t, K_{t-1}) \mapsto F(K_{t-1}, \chi^{\frac{\varphi}{1+\varphi}} W_t^{\frac{\varphi}{1+\varphi}})$ is a concave as the the composition of concave and increasing functions. We thus deduce that the mapping defined by $(W_t, K_{t-1}, K_t) \mapsto \log(W_t) + \log\left(K_{t-1} + F(K_{t-1}, \chi^{\frac{\varphi}{1+\varphi}} W_t^{\frac{\varphi}{1+\varphi}}) - \frac{1+\varphi+\varphi\beta}{(1+\beta)(1+\varphi)} W_t - K_t - G_t\right)$ is concave. Any interior optimum characterized by first-order conditions must be a maximum.

A.5 FOCs derivation

We focus on the case where unemployed agents are credit-constrained. Note that the situation where both unemployed agents are credit-constrained is not optimal whenever $u'(0) = \infty$. Indeed, when both agents are credit-constrained, deviating and having employed agents to save a small amount yields an finite increase in unemployed agents utility.

Using individual budget constraints, Euler equations (21) and (22) become:

$$u'\left(\frac{w_t(\chi w_t)^{\varphi}}{\varphi+1} - a_{e,t}\right) = \beta \mathbb{E}_t\left[R_{t+1} u'(R_{t+1} a_{e,t})\right], \tag{79}$$
$$u'(R_t a_{e,t-1}) > \beta \mathbb{E}_t\left[R_{t+1} u'\left(\frac{w_{t+1}(\chi w_{t+1})^{\varphi}}{\varphi+1} - a_{e,t+1}\right)\right].$$

Using log preferences, we deduce from Euler equation (79):

$$a_{e,t} = \frac{\beta}{1+\beta} \frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} \geq 0. \tag{80}$$

After some simplification, the Ramsey program can then be written as:

$$\max_{\{B_t,w_t,R_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log \left(\frac{1}{1+\beta} \frac{w_t(\chi w_t)^{\varphi}}{\varphi+1} \right) + \log(R_t \frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi}) \right), \tag{81}$$

$$w_{t+1}(\chi w_{t+1})^{\varphi} > \beta^2 R_{t+1} R_t w_t (\chi w_t)^{\varphi}, \tag{82}$$

$$\begin{aligned} G + B_{t-1} + (R_t - 1) \frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} + w_t(\chi w_t)^{\varphi} = B_t \\ + F(\frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} - B_{t-1}, (\chi w_t)^{\varphi}). \end{aligned} \tag{83}$$

Note that the Euler inequality for unemployed agents (82) is equivalent at the steady state to $\beta R < 1$, which will always hold in equilibrium.

The Lagrangian associated to program (81)–(83) can be written (up to some constants independent of policies):

$$\mathcal{L} = (1+\beta)(\varphi+1)\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log(w_t) + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log(R_t) + \log(a_{e,-1}) \tag{84}$$

$$\begin{aligned} &+ \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t \mu_t \Big(F(\frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} - B_{t-1}, (\chi w_t)^{\varphi}) + B_t - G_t - B_{t-1} \\ &\qquad - (R_t - 1) \frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} - w_t(\chi w_t)^{\varphi} \Big) \end{aligned} \tag{85}$$

$$+ \mu_0 \left(F(K_{-1}, (\chi w_0)^{\varphi}) + B_0 - G_0 - B_{-1} - (R_0 - 1) a_{-1} - w_0 (\chi w_0)^{\varphi} \right).$$

Defining by convention w_{-1} as $\frac{\beta}{1+\beta} \frac{w_{-1}(\chi w_{-1})^{\varphi}}{1+\varphi} = a_{-1}$, FOCs associated to the Lagrangian (84) can be summarized as (for $t \geq 0$):

$$\begin{aligned} 0 = (1+\beta)(\varphi+1) \frac{1}{w_t} + \beta(\chi w_t)^{\varphi} \frac{\beta}{1+\beta} \mathbb{E}_t \left[\mu_{t+1}(F_{K,t+1} - R_{t+1} + 1) \right] \\ + \chi \mu_t (\chi w_t)^{\varphi-1} \left(\varphi F_{L,t} - (\varphi+1) w_t \right), \end{aligned} \tag{86}$$

$$\mu_t = \beta \mathbb{E}_t \left[(1 + F_{K,t+1}) \mu_{t+1} \right], \tag{87}$$

$$1 = R_t \mu_t \frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi}. \tag{88}$$

We can take advantage of FOCs (87) and (88) to simplify FOC (86) as follows:

$$\mu_t w_t (\chi w_t)^{\varphi} \left(1 - (1+\beta)\varphi \frac{\tau_t^L}{1-\tau_t^L} \right) = (1+\varphi)(1+\beta), \tag{89}$$

which is a time-t equation only and does not raise convergence issues. The only dynamic FOC is the forward-looking equation (87). We will check that the system is well-defined and does not raise convergence issues.

A.6 Steady state and the Laffer Curve

Note that because of the FOC (88), $\mu = 0$ or $R = 0$ is not possible at the steady state. FOCs (86)–(88) and governmental budget constraint (83) become at the steady state, where we denote variable without subscripts:

$$\frac{1}{1+\beta}\mu w(\chi w)^{\varphi} = \varphi + 1 + \mu(\chi w)^{\varphi}\varphi(F_L - w), \tag{90}$$

$$1 = \beta(1 + F_K) \tag{91}$$

$$1 = R\mu\frac{\beta}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi} \tag{92}$$

$$F\Big(\frac{\beta}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi} - B, (\chi w)^{\varphi}\Big) = G + (R-1)\frac{\beta}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi} + w(\chi w)^{\varphi} \tag{93}$$

Using (92) and $w = (1-\tau^L)F_L$, equation (90) becomes:

$$\frac{1}{\beta} - R = \varphi\frac{1+\beta}{\beta}\left(\frac{F_L}{w} - 1\right). \tag{94}$$

Using $w = (1-\tau^L)F_L$, and $R - 1 = (1-\tau^K)F_K = (1-\tau^K)(\beta^{-1}-1)$, (94) yields:

$$\tau^K = \varphi\frac{1+\beta}{1-\beta}\frac{\tau^L}{1-\tau^L}. \tag{95}$$

After several manipulations and using (91) and (95), as well as the properties of F, the governmental budget constraint (93) implies that τ^L is a solution of the following equation:

$$\tau^L = \frac{1}{1-\alpha}\frac{\frac{G}{Y_{FB}}(1-\tau^L)^{-\varphi} - \overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}, \tag{96}$$

where $\overline{g}_1$ is defined in (75). Equation (96) can admit zero, one, or two solutions (as the right hand-side is convex). We now show it more formally : τ^L is thus a solution of the following equation:

$$\mathcal{T}(\tau^L) = 0, \tag{97}$$

$$\text{where: } \mathcal{T} : \tau \in (-\infty, 1) \mapsto \tau - \frac{1}{1-\alpha}\frac{\frac{G}{Y_{FB}}(1-\tau)^{-\varphi} - \overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}. \tag{98}$$

The mapping $\tau \mapsto \mathcal{T}(\tau)$ is akin to a Laffer curve. Indeed, we can check that $\mathcal{T}$ is continuously differentiable, strictly concave, with a unique maximum over $(-\infty, 1)$. In consequence, the function $\mathcal{T}$ admits either zero, one, or two solutions. The number of solutions depends on the level of public spending G in (98). When public spending is too high, there is no level of labor tax that make this public spending sustainable: $T(\tau) < 0$ for all $\tau \in (-\infty, 1)$. When the public spending is sustainable, T typically admits two roots. The smaller root corresponds to a low tax and a high labor supply, while the higher root corresponds to a high tax and a low labor supply. There is a third case that is the limit between sustainability and no sustainability. In this situation, there is a unique tax rate that enables public spending to be financed.

The limit case of the Laffer curve happens when the extremum point of the Laffer curve is the only root of the function. It can be checked that this corresponds to the tax level $\overline{\tau}^L_{La}$ that verifies $\mathcal{T}(\overline{\tau}^L_{La}) = \mathcal{T}'(\overline{\tau}^L_{La}) = 0$, or equivalently to:

$$\overline{\tau}^L_{La} = \frac{1}{1+\varphi} - \frac{1}{1-\alpha}\frac{\varphi}{1+\varphi}\frac{\overline{g}_1}{1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}+\frac{\varphi}{1+\varphi}}. \tag{99}$$

This corresponds to a ratio of public spending $\frac{G}{Y_{FB}}$, defined as:

$$\overline{g}_{La} := \frac{1-\alpha}{\varphi}\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}+\frac{\varphi}{1+\varphi}\right)(1-\overline{\tau}^L_{La})^{1+\varphi}. \tag{100}$$

So, any public spending such that $\frac{G}{Y_{FB}} > \overline{g}_{La}$ is not sustainable and cannot be financed by any tax system.

Regarding the allocation, we have:

$$c_e = \frac{1}{1+\beta}\frac{w(\chi w)^\varphi}{1+\varphi}, \tag{101}$$

$$c_u = \frac{1-(1-\beta)\tau^K}{1+\beta}\frac{w(\chi w)^\varphi}{1+\varphi}. \tag{102}$$

Finally, a condition for the $\tau^K > 0$-equilibrium to exist is $c_u > 0$, or equivalently, using (95) that the solution of (96) must verify:

$$(1+(1+\beta)\varphi)\tau^L < 1. \tag{103}$$

Oppositely, when $\frac{G}{Y_{FB}} < \overline{g}_{La}$, two different tax levels enable the government to finance public spending and the planner will always opt for the lowest tax rate. Indeed, taxes have an unambiguously negative impact on consumption levels, since they can be written as:

$$c_e = \frac{1}{1+\beta}(1-\tau^L)^{\varphi+1}\frac{w_{FB}(\chi w_{FB})^\varphi}{1+\varphi},\ c_u = (1-(1-\beta)\tau^K)c_e. \tag{104}$$

So larger taxes decrease consumption and hence individual welfare.

A.7 Non-existence of the $\tau^K = 0$-equilibrium

We prove here that the steady-state equilibrium featuring full risk-sharing and $\tau^K = 0$ does not exist. More precisely, we show that it is always dominated by the equilibrium featuring binding credit constraint and $\tau^K > 0$ (Sections A.5). We write with the 0-subscript relates the allocation where $\tau^K = 0$, and with no subscript the allocation where $\tau^K > 0$. The proof is split into two parts: (i) when the $\tau_k > 0$-equilibrium exists, i.e., when condition (103) holds (Section A.7.2); and (ii) when the $\tau_k > 0$-equilibrium does not exist, i.e., when condition (103) does not holds.

A.7.1 Characterization of the $\tau^K = 0$-equilibrium

We focus on the full insurance equilibrium with zero capital tax. As we use a 0 subscript to denote quantities in this case, we have $\tau_0^K = 0$. With the same steps as in Section A.2, we have:

$$w_0 = (1 - \tau^L) w_{FB}, \tag{105}$$

$$K_0 = (1 - \tau^L)^{\varphi} K_{FB}, \tag{106}$$

$$Y_0 = (1 - \tau^L)^{\varphi} Y_{FB}, \tag{107}$$

Governmental budget constraint (20) becomes:

$$B_0 = -\frac{\beta}{1-\beta} G + \frac{\beta}{1-\beta} \tau_0^L (1 - \tau_0^L)^{\varphi} w_{FB} (\chi w_{FB})^{\varphi}.$$

Perfect risk sharing (i.e., $c_{u,0} = c_{e,0} - \frac{1}{\chi} \frac{l_{e,0}^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}$) and financial market clearing (i.e., $A_0 = K_0 + B_0$) imply (as in (72) and (72)), after proper substitution:

$$a_{e,0} - a_{u,0} = \frac{\beta}{1+\beta} \frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi + 1} (1 - \tau_0^L)^{\varphi+1}, \tag{108}$$

$$a_{u,0} + a_{e,0} = (1 - \tau^L)^{\varphi} K_{FB} - \frac{\beta}{1-\beta} G + \frac{\beta}{1-\beta} \tau_0^L (1 - \tau_0^L)^{\varphi} w_{FB} (\chi w_{FB})^{\varphi}. \tag{109}$$

We deduce by combination of the two previous equations:

$$\begin{aligned} 2a_{u,0} = {} & (1 - \tau_0^L)^{\varphi} K_{FB} - \frac{\beta}{1-\beta} G - \frac{\beta}{1+\beta} \frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1} (1 - \tau_0^L)^{\varphi+1} \\ & + \frac{\beta}{1-\beta} \tau_0^L (1 - \tau_0^L)^{\varphi} w_{FB} (\chi w_{FB})^{\varphi}. \end{aligned}$$

Dividing by Y_0 of (107) and using notation (69)–(71) and (25), we obtain:

$$2\frac{a_{u,0}}{Y_0} = \frac{\beta}{1-\beta} (\overline{g}_1 - g_{FB}(1 - \tau_0^L)^{-\varphi}) + \left(\frac{1}{1-\beta} + \frac{1}{1+\beta} \frac{1}{\varphi+1} \right) \beta \tau_0^L (1 - \alpha). \tag{110}$$

We turn to the computation of $a_{e,0}$. Using (108) and (109), we get:

$$2\frac{a_{e,0}}{Y} = 2\frac{a_{u,0}}{Y} + 2\frac{\beta}{1+\beta} \frac{1-\alpha}{\varphi+1} (1 - \tau_0^L),$$

implying that $a_{e,0} \geq a_{u,0}$ for all values of $\tau_0^L \leq 1$. We compute the consumption level $c_{u,0}$ from individual budget constraint (17):

$$2\frac{c_{u,0}}{Y_{FB}} = (1-\tau_0^L)^{\varphi}\overline{g}_1 - \frac{G}{Y_{FB}} + \frac{2}{1+\beta}\frac{1-\alpha}{\varphi+1}(1-\tau_0^L)^{\varphi} + \frac{\varphi}{\varphi+1}(1-\alpha)\tau_0^L(1-\tau_0^L)^{\varphi}. \tag{111}$$

Computing the derivative of $2\frac{c_{u,0}}{Y_{FB}}$ with respect to the labor tax τ_0^L yields:

$$\frac{1}{\varphi(1-\tau_0^L)^{\varphi-1}}\frac{\partial}{\partial\tau_0^L}2\frac{c_{u,0}}{Y_{FB}} = -\frac{(1-\beta)\alpha}{1+\beta(\delta-1)} - (1-\alpha)\tau_0^L < 0, \tag{112}$$

whenever $\tau_0^L \geq 0$. We deduce from the last inequality that $c_{u,0}$ (and hence aggregate welfare since $c_{u,0} = c_{e,0} - \frac{1}{\chi}\frac{l_0^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}$) is decreasing with τ_0^L. Since $a_{e,0} \geq a_{u,0}$ for all values of τ_0^L, the value of τ_0^L is chosen as small as possible for credit constraints not to bind and hence such that $a_{u,0} = 0$. From (110), τ_0^L is the solution of:

$$\tau_0^L = \frac{1}{1+\frac{1-\beta}{1+\beta}\frac{1}{\varphi+1}}\frac{g_{FB}(1-\tau_0^L)^{-\varphi} - \overline{g}_1}{1-\alpha}. \tag{113}$$

In words, the planner chooses the lowest possible labor tax to reduce distortions. Finally, regarding allocation, we compute:

$$c_{u,0} = c_{e,0} - \chi^{-1}\frac{l_0^{1+1/\varphi}}{1+1/\varphi} = \frac{1}{1+\beta}\frac{w_0(\chi w_0)^{\varphi}}{1+\varphi}. \tag{114}$$

Laffer curve. Equation (113) admits 0, 1 or 2 solutions, and reflects some form of Laffer curve. The case with zero solution appears when no equilibrium exists: the public spending G is too high to be financed and no level of labor tax allows the governmental budget to hold. The case with 2 solutions is the standard case when the equilibrium exists: it features either a low tax/high labor supply or a high tax/low labor supply combination. The planner (since inequality (112) holds) unambiguously opts for the lowest tax. Finally the 1-solution case is a limit case that only occurs for a unique value of public spending.

A.7.2 The program of the planner

The fact that the planner implements $a_{u,0} = 0$ in the equilibrium with full rik-sharing implies that the objective of the planner is actually the same in the case with binding credit constraints, provided in the program (28). As a consequence, the allocation with $\tau^K = 0$ and $\tau^K > 0$ can be written as outcome of the same program, with the constraint $\tau^K \geq 0$.

More formally, we consider the following program:

$$\max_{\{B_t,w_t,R_t\}} \sum_{t=0}^{\infty} \beta^t \left((1+\beta)\log\left(\frac{1}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{\varphi+1}\right) + \log(\beta R_t)\right) \tag{115}$$

$$G + B_{t-1} + (R_t - 1)\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} + w_t(\chi w_t)^{\varphi} = B_t \tag{116}$$
$$+ F\left(\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} - B_{t-1}, (\chi w_t)^{\varphi}\right),$$
$$R_t \geq 1 + \tilde{r}_t \tag{117}$$

where the interest rate $\tilde{r}_t$ in the constraint (117) is taken as exogenous with $\tilde{r}_t = F_{K,t}$. We now show that the previous program has the desired properties.

We start with the case $\tau^K = 0$. Denoting by $\beta^t \mu_t$ the Lagrange multiplier associated to the constraint (116), the maximization with respect to B_t yields: $\mu_t = \beta(1 + F_{K,t+1})\mu_{t+1}$,or at the steady state: $\beta(1 + F_K) = 1$.The constraint (116) implies then at the steady state, using (68)–(71) that the labor tax, denoted $\hat{\tau}_0^l$ verifies:

$$(1-\alpha)\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)\hat{\tau}_0^l = \frac{g_{FB}}{(1-\hat{\tau}_0^l)^{\varphi}} - \bar{g}_1, \tag{118}$$

which is the equation as (113) for τ_0^L. Since the planner will also choose the lowest solution to (118), we deduce that $\hat{\tau}_0^l = \tau_0^L$. Consumption levels then mechanically verify equation (114), which proves that the steady-state equilibrium with $\tau^K = 0$ is a steady-state solution of the program (115)–(116) where we impose $\tau_t^K = 0$ at all dates.

We now turn to the unconstrained case ($\tau^K \neq 0$). In that case, the FOCs of the program (115)–(116), with respect to B_t, R_t, and w_t, respectively are:

$$\mu_t = \mu_{t+1}\beta(1 + F_{K,t}),$$
$$1 = R_t \mu_t \frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi},$$
$$\frac{(1+\beta)(1+\varphi)}{w_t} = \frac{\mu_t}{w_t}\left((\varphi+1)w_t(\chi w_t)^{\varphi} - \varphi F_{L,t}(\chi w_t)^{\varphi}\right)$$
$$+ \frac{\beta\mu_{t+1}}{w_t}(R_{t+1} - 1 - F_{K,t+1})\frac{\beta}{1+\beta}w_t(\chi w_t)^{\varphi}.$$

At the steady-state, we obtain:

$$1 = \beta(1 + F_K), \tag{119}$$

$$1 = R\mu\frac{\beta}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi}, \tag{120}$$

$$\frac{(1+\beta)(1+\varphi)}{\mu(\chi w)^{\varphi}} = (\varphi+1)w - \varphi F_L + \beta(R - 1 - F_K)\frac{\beta}{1+\beta}w. \tag{121}$$

With (119) and (120), equation (121) yields, after some manipulation that taxes $\hat{\tau}^k$ and $\hat{\tau}^l$ verify:

$$\hat{\tau}^k = \varphi \frac{1+\beta}{1-\beta} \frac{\hat{\tau}^l}{1-\hat{\tau}^l},$$

which is the same relationship as (95) for τ^K. As we did in the constrained case, the constraint (116) of the program at the steady state yields for $\hat{\tau}^l$ the same definition as equation (96) for τ^L. We deduce that $\hat{\tau}^l = \tau^L$ and $\hat{\tau}^k = \tau^K$, when τ^L satisfies condition (103). Consumption levels (101) and (102) then easily follow. It is also easy to check that $\tau^K, \tau^L > 0$.

We therefore deduce that the allocation with $\tau^K = 0$ is the solution of a constrained program and is hence dominated by the allocation $\tau_k \neq 0$ – when ever the later exist.[14]

A.8 A non-interior steady-state equilibrium

Here we investigate the case when (96) admits a solution, but when this solution does not verify condition (103). We have:

$$\left(1 - (1+\varphi(1+\beta))\tau_t^L\right)(1-\tau_t^L)^{\varphi}\mu_t \tilde{w}_t (\chi \tilde{w}_t)^{\varphi} = (1+\beta)(1+\varphi), \tag{122}$$

$$\frac{\mu_{t+1}}{\mu_t} = \frac{1}{\beta(1+F_{K,t+1})}, \tag{123}$$

$$(1+(1-\tau_t^K)F_{K,t})\mu_t(1-\tau_{t-1}^L)^{\varphi+1}\tilde{w}_{t-1}(\chi\tilde{w}_{t-1})^{\varphi} = \frac{(1+\beta)(1+\varphi)}{\beta} \tag{124}$$

Equation (122) implies that for all t:

$$\tau_t^L \leq \frac{1}{1+\varphi(1+\beta)}.$$

In particular, $\tau^L = \lim_{t\to\infty} \tau_t^L \leq \frac{1}{1+\varphi(1+\beta)}$. From (122), we also understand that there are possibly non-interior steady states, featuring $\lim_t \mu_t = \infty$ or $\lim_t \tilde{w}_t = \infty$.

First case: $\lim w_t = w^* < \infty$.

– The case $w^* = 0$ is not possible. Otherwise there are no resources to pay G.

– Assume that $\lim \mu_t = \infty$, then equation (122) implies $\lim_t \tau_t^L = (1+\varphi(1+\beta))^{-1}$. Equation (124) then yields $\lim_t (1+(1-\tau_t^K)F_{K,t}) = \lim_t R_t = 0$.

[14] Note that the argument could not be applied right away from the initial program formulation of Section 3 because with $\tau_k \neq 0$, the constraint $a_{u,t} = 0$ was binding – which is not present anymore with the modified program (115)–(116).

Second case: $\lim_t w_t = \infty$. We thus have $\lim_t \tilde{w}_t = \infty$. We also have from factor price definitions:

$$\chi\tilde{w}_t = \left(\frac{\chi(1-\alpha)}{(1-\tau_t^L)^{\alpha\varphi}}\right)^{\frac{1}{1+\varphi\alpha}} K_{t-1}^{\frac{\alpha}{1+\varphi\alpha}},$$

which yields $\lim K_t = \infty$ and $\lim_t \frac{K_{t-1}}{(\chi w_t)^\varphi} = \infty$. We deduce $\lim_t F_{K,t} = -\delta$. We then deduce $\lim_t \mu_t = \infty$, $\lim_t \tau_t^L = (1+\varphi(1+\beta))^{-1}$, and $\lim_t R_t = 0$.

These two non-stationary equilibria feature $\lim_t \mu_t = \infty$ and $\lim_t R_t = 0$.

A.9 Characterization of positive public debt

The financial market clearing condition (19) implies using (80) and the definition of w:

$$B = (\chi w)^\varphi \left(\frac{\beta}{1+\beta}\frac{1-\tau^L}{1+\varphi}F_L - \frac{K}{L}\right),$$

which is positive iff: $\frac{\beta}{1+\beta}\frac{1-\tau^L}{1+\varphi} > \frac{1}{F_L}\frac{K}{L}$. Using FOC (156) and the definitions of F and $\overline{g}_1$, we can simplify $\frac{1}{F_L}\frac{K}{L}$ and obtain that $B > 0$ iff:

$$\tau^L < -\frac{1+\varphi}{1-\alpha}\frac{1+\beta}{1-\beta}\overline{g}_1. \tag{125}$$

Using the expression (96) of τ^L, we get an equivalent condition to (125):

$$g_{FB}(1-\tau^L)^{-\varphi} < \overline{g}_{\text{pos}},$$
$$\text{where: } \overline{g}_{\text{pos}} = \frac{1+\beta}{1-\beta}(1+2\varphi)(-\overline{g}_1). \tag{126}$$

A.10 Model dynamics in the presence of aggregate shocks

A.10.1 Model linearization

Defining:

$$\theta = \frac{1}{1+\varphi}\frac{\beta}{1+\beta}, \tag{127}$$

FOCs (86) and (87) and governmental budget constraint (83) become:

$$\mu_t = \beta(1+\alpha Z_{t+1}K_t^{\alpha-1}\chi^{(1-\alpha)\varphi}w_{t+1}^{(1-\alpha)\varphi} - \delta)\mu_{t+1}, \tag{128}$$

$$0 = 1 - \mu_t w_t(\chi w_t)^\varphi(1-\theta) + \frac{\varphi}{1+\varphi}\mu_t(1-\alpha)K_{t-1}^\alpha(\chi w_t)^{\varphi(1-\alpha)}, \tag{129}$$

$$K_{t-1}^\alpha(\chi w_t)^{\varphi(1-\alpha)} = G_t + K_t - (1-\delta)K_{t-1} + \frac{1}{\mu_t} + (1-\theta)w_t(\chi w_t)^\varphi. \tag{130}$$

We deduce R_t from $1 = R_t\mu_t\theta w_{t-1}(\chi w_{t-1})^{\varphi}$ (i.e., FOC (88)) and B_t from $B_t = \theta w_t(\chi w_t)^{\varphi} - K_t$ (i.e., financial market clearing).

We denote by a hat the proportional deviation to the steady state value. Formally, for a generic variable x: $\widehat{x} = \frac{x_t - x}{x}$. The linearization of equations (128)–(130) yield after some manipulation:

$$\widehat{\mu}_t - E_t\widehat{\mu}_{t+1} = (1-\beta(1-\delta))(\widehat{Z}_{t+1} + (\alpha-1)\widehat{K}_t + (1-\alpha)\varphi E_t\widehat{w}_{t+1}), \tag{131}$$

$$0 = -\alpha\widehat{K}_{t-1} + (A-1)\widehat{\mu}_t + ((\varphi+1)(A-1)+1+\varphi\alpha)\widehat{w}_t, \tag{132}$$

$$0 = \frac{G}{Y}\widehat{G}_t + \frac{\alpha}{\frac{1}{\beta}-(1-\delta)}\left(\widehat{K}_t - \beta^{-1}\widehat{K}_{t-1}\right) - (A-1)\varphi\frac{1-\alpha}{1+\varphi}\widehat{\mu}_t + (A-1)\,\varphi(1-\alpha)\widehat{w}_t, \tag{133}$$

where τ^L is defined in (96) and where:

$$A = (1+\frac{1}{\varphi(1+\beta)})(1-\tau^L) > 1, \tag{134}$$

where the inequality comes from condition (103) for the existence of the equilibrium.

A.10.2 Public debt spending shock

In the remainder, we will focus on full capital depreciation: $\delta = 1$.

Dynamic system. In that case, we can show that, when setting:

$$r_\mu = \frac{(1+\varphi)(A-1)+1+\alpha\varphi}{(1+\alpha\varphi)A}, \tag{135}$$

$$t_\mu = (1-\alpha)\frac{(1+\varphi)(A-1)+1}{(1+\alpha\varphi)A}, \tag{136}$$

$$r_K = \frac{1-\alpha}{\alpha\beta}(A-1)\frac{\varphi}{1+\varphi}\left(1+\frac{(1+\varphi)(A-1)}{(1+\varphi)(A-1)+1+\varphi\alpha}\right), \tag{137}$$

$$t_K = \frac{1}{\beta}\frac{(1+\varphi\alpha)A}{(1+\varphi)(A-1)+1+\varphi\alpha}, \tag{138}$$

$$s_K = -\frac{G}{\alpha\beta Y}, \tag{139}$$

we obtain from (131)–(133):

$$E_t\left[\widehat{\mu}_{t+1}\right] = r_\mu\widehat{\mu}_t + t_\mu\widehat{K}_t, \tag{140}$$

$$\widehat{K}_t = r_K\widehat{\mu}_t + t_K\widehat{K}_{t-1} + s_K\widehat{G}_t, \tag{141}$$

where the dynamics of $\widehat{G}_t$ is given by:

$$\widehat{G}_t = \rho_G \widehat{G}_{t-1} + \sigma_G \varepsilon_{G,t}, \tag{142}$$
$$\text{where: } \varepsilon_{G,t} \sim_{\text{IID}} \mathcal{N}(0,1),$$

and $\sigma_G > 0$ and $\rho_G \in (-1, 1)$.

Since $A > 1$, it can be checked that the coefficients t_K, r_K, t_μ are positive, while $r_\mu > 1$.

Deriving a simplified dynamic system. We look for coefficients coefficients ρ_K, σ_K, ρ_μ, σ_μ, such that:

$$\widehat{K}_t = \rho_K \widehat{K}_{t-1} + \sigma_K \widehat{G}_t \tag{143}$$
$$\widehat{\mu}_t = \rho_\mu \widehat{K}_{t-1} + \sigma_\mu \widehat{G}_t \tag{144}$$

Combining (140)–(141) yields:

$$\begin{aligned} E_t \widehat{K}_{t+1} &= r_\mu(\widehat{K}_t - t_K \widehat{K}_{t-1} - s_K \widehat{G}_t) + r_K t_\mu \widehat{K}_t + t_K \widehat{K}_t + s_K \rho_G \widehat{G}_t \\ &= -r_\mu t_K \widehat{K}_{t-1} - s_K r_\mu \widehat{G}_t + (r_K t_\mu + r_\mu + t_K)\widehat{K}_t + s_K \rho_G \widehat{G}_t \end{aligned}$$

$$E_t \widehat{K}_{t+1} - (t_K + r_\mu + r_K t_\mu)\widehat{K}_t + r_\mu t_K \widehat{K}_{t-1} = (s_K \rho_G - r_\mu s_K)\widehat{G}_t.$$

Using (143), we obtain that ρ_K must verify solve the following equation:

$$\rho_K^2 - (t_K + r_\mu + r_K t_\mu)\rho_K + r_\mu t_K = 0, \tag{145}$$

whose discriminant is:

$$D = (t_K + r_\mu + r_K t_\mu)^2 - 4 r_\mu t_K. \tag{146}$$

Since $t_K, r_\mu, r_K, t_\mu \geq 0$, we have $D \geq (t_K + r_\mu)^2 - 4r_\mu t_K = (t_K - r_\mu)^2 > 0$, where the strict inequality comes from $t_K = \frac{1}{\beta r_\mu} > 0$. Equation (145) thus admits two distinct roots, which are:

$$\rho_{K,1} = \frac{t_K + r_\mu + r_K t_\mu + \sqrt{D}}{2} \text{ and } \rho_{K,2} = \frac{t_K + r_\mu + r_K t_\mu - \sqrt{D}}{2}. \tag{147}$$

Since $(t_K + r_\mu + r_K t_\mu)^2 > D > 0$, we deduce that $0 < \rho_{K,2} < \rho_{K,1}$. Furthermore, we can check that a necessary and sufficient condition for the equilibrium to be stable is:

$$\alpha \leq \frac{1}{1 + (1-\beta)(1+\varphi)} < 1, \tag{148}$$

where the second inequality comes from $\beta \in (0, 1)$. Note that a sufficient condition for the stability is $\overline{g}_1 < 0$ – which is equivalent to $\alpha \leq \frac{1}{1+(1+\beta)(1+\varphi)}$ and hence implies (148).

Let us prove it. The condition $\rho_{K,2} < 1$ is equivalent to $J := t_K + r_\mu + r_K t_\mu - r_\mu t_K - 1 > 0$. Using equations (135)–(138), we can show that:

$$\begin{aligned}\frac{J}{J_0} &= (\beta(1+\varphi)(A-1) + (1+\alpha\varphi)(\beta - A)) \\ &+ \frac{1-\alpha}{\alpha(1+\varphi)}((1+\varphi)(A-1)+1)\left(2(1+\varphi)(A-1)+1+\varphi\alpha\right), \\ \text{where: } J_0 &= \frac{\varphi(1-\alpha)(A-1)}{\beta(1+\alpha\varphi)A((1+\varphi)(A-1)+1+\varphi\alpha)}.\end{aligned}$$

Since $A > 1$, $J_0 > 0$ and the sign of J is the one of:

$$\beta(1+\varphi)(A-1) + (1+\alpha\varphi)(\beta - 1 - (A-1)) + \\ \frac{1-\alpha}{\alpha(1+\varphi)}((1+\varphi)(A-1)+1)\left(2(1+\varphi)(A-1)+1+\varphi\alpha\right),$$

which can be seen as a quadratic polynomial in $A-1$, that we denote $P(\cdot)$. After some rearrangement, we obtain:

$$\begin{aligned}P(A-1) &= \frac{1+\alpha\varphi}{1+\varphi}(-(1-\beta)(1+\varphi) + \frac{1-\alpha}{\alpha}) + \\ &+ (A-1)\left(-(1-\beta)(1+\varphi) + \frac{1-\alpha}{\alpha} + 2(1+\alpha\varphi)\frac{1-\alpha}{\alpha}\right) \\ &+ (A-1)^2\frac{1-\alpha}{\alpha}2(1+\varphi).\end{aligned}$$

A necessary condition for $P(A-1) > 0$ for all $A > 1$ is $P(0) \geq 0$. However, $P(0) \geq 0 \Rightarrow P'(0) > 0$ (since $\beta \in (0,1)$). So, since $P''(0) \geq 0$, $P(0) \geq 0$ is a necessary and sufficient condition for $P(A-1) > 0$ for $A > 1$. The condition $P(0) \geq 0$ is equivalent to condition (148), which concludes the proof regarding equilibrium stability.

Stability and characterization of the system (143)–(144). The Blanchard-Kahn conditions involve checking that $\rho_K < 1$.

Since $0 < \rho_{K,2} < \rho_{K,1}$ and $\rho_{K,2}\rho_{K,1} = \beta^{-1} > 1$, we must have $\rho_{K,1} > 1$, which imposes that $\rho_K = \rho_{K,2}$. The stability Blanchard-Kahn condition requires $\rho_{K,2} < 1$. Note that in the limit case when the equilibrium does not exist (i.e., condition (103) holds with equality), and which corresponds to $A = 1$, it is straightforward to check that $\rho_{K,2} = 1$ and that the dynamic system is not stable.

To characterize further the dynamic system (143)–(144), we deduce from (140)–(141) that ρ_μ is connected through ρ_K with:

$$(r_\mu - \rho_K)\rho_\mu = -t_\mu \rho_K. \tag{149}$$

Since $r_\mu > 1$, $t_\mu > 0$, and $\rho_K \in (0,1)$, we deduce that $\rho_\mu < 0$.

Regarding parameters σ_K and σ_μ, we have from (140)–(141):

$$\sigma_K = r_K \sigma_\mu + s_K, \tag{150}$$

$$r_\mu \sigma_\mu = (\rho_\mu - t_\mu)\sigma_K + \sigma_\mu \rho_G. \tag{151}$$

Equation (151) implies:

$$(r_\mu - \rho_G)\sigma_\mu = (\rho_\mu - t_\mu)\sigma_K. \tag{152}$$

Using $r_\mu > 1 > \rho_G$ and (149) implying that $\rho_\mu - t_\mu = r_\mu \rho_\mu / \rho_K < 0$, we deduce that σ_μ and σ_K have opposite signs. Using $r_K > 0$ and $s_K < 0$ in equation (150), we deduce that $\sigma_\mu > 0 > \sigma_K$.

The role of the shock persistence ρ_G. Combining (150) and (151) yields:

$$(r_\mu + (t_\mu - \rho_\mu) r_K)\sigma_\mu = (\rho_\mu - t_\mu) s_K + \sigma_\mu \rho_G,$$

which yields, by the implicit function theorem:

$$(r_\mu - \rho_G + (t_\mu - \rho_\mu) r_K)\frac{\partial \sigma_\mu}{\partial \rho_G} = \sigma_\mu,$$

since only σ_μ (and σ_K) depend on ρ_G. Since $r_\mu > 1 > \rho_G$, and $\sigma_\mu, t_\mu, r_K > 0 > \rho_\mu$, we deduce using the previous equation and (150) that:

$$\frac{\partial \sigma_\mu}{\partial \rho_G} > 0 \text{ and } \frac{\partial \sigma_K}{\partial \rho_G} > 0.$$

The previous derivative, and equation (144) impliying $\widehat{\mu}_0 = \sigma_\mu \widehat{G}_0$ implies that for the same initial shock $\widehat{G}_0$, the increase in $\widehat{\mu}_0$ is higher , the higher the persistence

$$\left.\frac{\partial \widehat{\mu}_0}{\partial \rho_G}\right|_{\widehat{G}_0} > 0 \tag{153}$$

Then, from (132), we have

$$\widehat{w}_0 = -\frac{A-1}{((\varphi+1)(A-1)+1+\varphi\alpha}\widehat{\mu}_0 \tag{154}$$

which implies $\left.\frac{\partial \widehat{w}_0}{\partial \rho_G}\right|_{\widehat{G}_0} < 0$. Finally, from and $\frac{\partial \sigma_K}{\partial \rho_G} > 0$,we deduce $\frac{\partial \widehat{K}_0}{\partial \rho_G} < 0$.

Dynamic of the capital stock. By induction we can then prove that the dynamics (142) and (143) of $\widehat{G}_t$ and $\widehat{K_t}$ can be written as:

$$\widehat{G}_t = \rho_G^t \sigma_G \varepsilon_{G,0},$$

$$\widehat{K_t} = \sigma_K \sigma_G \frac{\rho_G^{t+1} - \rho_K^{t+1}}{\rho_G - \rho_K}\varepsilon_{G,0},$$

where by assumption we have $\widehat{K}_{-1} = \widehat{G}_{-1} = 0$ (no deviation from the steady state).

Let define:

$$\phi(t) = \begin{cases} \frac{\rho_K^{t+1} - \rho_G^{t+1}}{\rho_K - \rho_G} & \text{if } \rho_K \neq \rho_G, \\ (t+1)\rho_G^t & \text{if } \rho_K = \rho_G, \end{cases}$$

with $\phi(0) = 1$, $\phi(\infty) = 0$, and:

$$(\rho_K - \rho_G)\phi'(t) = \ln(\rho_K)\rho_K^{t+1} - \ln(\rho_G)\rho_G^{t+1}.$$

We have $\phi'(t_m) = 0$ iff:

$$t_m + 1 = \begin{cases} \frac{\ln(-\ln(\rho_K)) - \ln(-\ln(\rho_G))}{\ln(\rho_G) - \ln(\rho_K)} > 0 & \text{if } \rho_K \neq \rho_G, \\ -\frac{1}{\ln(\rho_G)} > 0 & \text{if } \rho_K = \rho_G, \end{cases}$$

It is direct to check that $\phi'(t) > 0$ iff $t < t_m$. The capital response is procyclical (it has the sign of $\widehat{G}_0$). When $\widehat{G}_0 > 0$, capital increases until date t_m before decreasing and converging back to its steady-state value.

We now investigate the impact of ρ_G on t_m. Defining $r_G := -\ln(\rho_G)$ and $r_K := -\ln(\rho_K)$, we obtain:

$$\frac{\partial t_m}{\partial r_G} = \frac{\frac{r_G - r_K}{r_G} - (\ln(r_G) - \ln(r_K))}{(r_G - r_K)^2} \text{ if } \rho_K \neq \rho_G.$$

By Taylor-Lagrange theorem, there exists $r \in (r_K, r_G)$, such that:

$$\ln(r_K) - \ln(r_G) = \frac{r_K - r_G}{r_G} - \frac{(r_K - r_G)^2}{2r^2},$$

from which we deduce:

$$\frac{\partial t_m}{\partial r_G} = \begin{cases} \frac{-\frac{(r_K - r_G)^2}{2r^2}}{(r_G - r_K)^2} < 0 & \text{if } \rho_K \neq \rho_G, \\ -\frac{1}{r_G^2} < 0 & \text{if } \rho_K = \rho_G, \end{cases}$$

So t_m decreases with r_G and increases with ρ_G: the more persistent ρ_G, the longer the impact of capital dynamics.

We now study the impact of ρ_G on the $\phi(t_m)$, the maximal value of ϕ (which corresponds to the maximal variation of capital stock following the public spending shock.

$$\phi(t_m) = \begin{cases} \frac{e^{-\frac{\ln(r_G) - \ln(r_K)}{r_G - r_K} r_K} - e^{-\frac{\ln(r_G) - \ln(r_K)}{r_G - r_K} r_G}}{e^{-r_K} - e^{-r_G}} & \text{if } \rho_K \neq \rho_G, \\ r_G^{-1} e^{-r_G(r_G^{-1} - 1)} > 0 & \text{if } \rho_K = \rho_G, \end{cases}$$

We focus on the case where $\rho_K \neq \rho_G$ and $\rho_K < 1$. Note that we have:

$$\phi(t_m) \to_{\rho_G \to 1} \frac{1}{1-\rho_K}.$$

and

$$\begin{aligned}\phi(t_m) &= \frac{\left(\frac{r_K}{r_G}\right)^{\frac{r_K}{r_G - r_K}} - \left(\frac{r_K}{r_G}\right)^{\frac{r_G}{r_G - r_K}}}{e^{-r_K} - e^{-r_G}} \\ &= \frac{\left(\frac{r_G}{r_K}\right)^{-\frac{1}{r_G/r_K - 1}} - \left(\frac{r_G}{r_K}\right)^{-\frac{1}{r_G/r_K - 1} - 1}}{e^{-r_K}\left(1 - e^{-r_K(r_G/r_K - 1)}\right)}\end{aligned}$$

We define $x := r_G/r_K - 1$, such that $\frac{r_G}{r_K} = 1 + x$, $\frac{r_K}{r_G - r_K} = \frac{1}{r_G/r_K - 1} = \frac{1}{x}$, and $\frac{r_G}{r_G - r_K} = 1 + \frac{1}{x}$, and we define $f(x) := \phi(t_m) = \frac{(1+x)^{-\frac{1}{x}} - (1+x)^{-\frac{1}{x} - 1}}{1 - e^{-r_K x}}$, such that:

$$(1+x)^{\frac{1}{x}+1} f'(x) = \frac{\frac{\ln(1+x)}{x}(1 - e^{-r_K x}) - x r_K e^{-r_K x}}{(1 - e^{-r_K x})^2}.$$

Note that:

$$(1+x)^{\frac{1}{x}+1} f'(x) \sim_{x \to -1} \frac{\ln(1+x)}{e^{r_K} - 1},$$

which is negative whenever x is sufficiently close to -1. In other words, f decreases with $x = r_G/r_K - 1$, and hence increases with ρ_G.

Dynamic of public debt. Regarding public debt, the financial market clearing implies that $B_t = \frac{\beta}{1+\beta} \frac{\chi^\varphi}{1+\varphi} w_t^{1+\varphi} - K_t$ and thus that the dynamics is given by:

$$B \widehat{B}_t = \frac{\beta}{1+\beta} \chi^\varphi w^{1+\varphi} \widehat{w}_t - K \widehat{K}_t.$$

At impact ($t = 0$), we have:

$$B \widehat{B}_0 = -\left(\frac{\beta}{1+\beta} \chi^\varphi w^{1+\varphi} \frac{A-1}{(\varphi+1)(A-1) + 1 + \varphi\alpha} \sigma_\mu + \sigma_K K \right) \sigma_G \varepsilon_{G,0} \tag{155}$$

As a consequence, if the public debt is positive at the steady state ($B > 0$ equivalent to $\bar{g}_1 < 0$ – see Section A.9, then for a positive initial shock, $\varepsilon_{G,0} > 0$, $\frac{\partial \sigma_K}{\partial \rho_G} > 0$ implies $\frac{\partial \widehat{B}_0}{\partial \rho_G} < 0$. The higher the shock persistence, the variation of public debt at impact decreases.

$$\frac{\partial \widehat{B}_0}{\partial \rho_G} < 0.$$

Using (152) with (155) and FOC (156) to simplify $\frac{K}{L F_L}$ into $\frac{\beta\alpha}{1-\alpha}$, we obtain:

$$B \widehat{B}_0 = -\sigma_\mu \sigma_G (\chi w)^\varphi F_L \left(\frac{\beta}{1+\beta} \frac{(1-\tau^L)(A-1)}{(\varphi+1)(A-1) + 1 + \varphi\alpha} + \frac{r_\mu - \rho_G}{\rho_\mu - t_\mu} \frac{\beta\alpha}{1-\alpha} \right) \varepsilon_{G,0},$$

and finally using the relationship (134) between A and $1-\tau^L$:

$$B\widehat{B}_0 = -\frac{\sigma_\mu \sigma_G (\chi w)^\varphi F_L \beta}{\rho_\mu - t_\mu}\Big(\frac{\varphi}{1+\varphi(1+\beta)}\frac{A(A-1)(\rho_\mu - t_\mu)}{(\varphi+1)(A-1)+1+\varphi\alpha} + (r_\mu - \rho_G)\frac{\alpha}{1-\alpha}\Big)\varepsilon_{G,0}, \quad (156)$$

Even if public debt is positive at the steady state ($B>0$), the sign of $\widehat{B}_0$ is ambiguous, since $\rho_\mu - t_\mu < 0$. It is the same for the quantity between brackets in (156) that can be positive or negative, depending in particular on the magnitude of the persistence ρ_G of the public spending shock. We illustrate it below in a particular tractable case.

B First-order conditions of the individual Ramsey program

The Ramsey problem can be written as follows:

$$\max_{\left(r_t, \tilde{w}_t, \tilde{r}_t, \tau_t^K, \tau_t, \kappa_t, B_t, K_t, L_t, (a_t^i, c_t^i, l_t^i, \nu_t^i)_i\right)_{t\geq 0}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t \int_i \omega_t^i U(c_t^i, l_t^i)\ell(di)\right], \quad (157)$$

$$(158)$$

$$G_t + R_t B_{t-1} + (R_t - 1)K_{t-1} + w_t\int_i (y_t^i l_t^i)^{1-\tau_t}\ell(di) = K_{t-1}^\alpha L_t^{1-\alpha} - \delta K_{t-1} + B_t \quad (159)$$

$$\text{for all } i\in\mathcal{I}\text{: } a_t^i + c_t^i = R_t a_{t-1}^i + w_t (y_t^i l_t^i)^{1-\tau_t}, \quad (160)$$

$$a_t^i \geq -\bar{a},\ \nu_t^i(a_t^i + \bar{a}) = 0,\ \nu_t^i \geq 0, \quad (161)$$

$$U_c(c_t^i, l_t^i) = \beta\mathbb{E}_t\left[R_{t+1}U_c(c_{t+1}^i, l_{t+1}^i)\right] + \nu_t^i, \quad (162)$$

$$-U_l(c_t^i, l_t^i) = (1-\tau_t)w_t(y_t^i)^{1-\tau_t}(l_t^i)^{-\tau_t}U_c(c_t^i, l_t^i), \quad (163)$$

$$K_t + B_t = \int_i a_t^i\ell(di),\ L_t = \int_i y_t^i l_t^i \ell\,(di)\,. \quad (164)$$

The Lagrangian can be written as:

$$\begin{aligned}
\mathcal{L} = {} & \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\int_i \omega_t^i U(c_t^i, l_t^i)\ell(di) && (165)\\
& - \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\int_i \left(\lambda_{c,t}^i - R_t\lambda_{c,t-1}^i\right)U_c(c_t^i, l_t^i)\ell(di)\\
& + \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\int_i \lambda_{l,t}^i\left(U_l(c_t^i, l_t^i) + (1-\tau_t)w_t(y_t^i)^{1-\tau_t}(l_t^i)^{-\tau_t}U_c(c_t^i, l_t^i)\right)\ell(di)\\
& - \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mu_t\Big(G_t + (1-\delta)B_{t-1} + (R_t - 1 + \delta)\int_i a_{t-1}^i\ell(di) + w_t\int_i (y_t^i l_t^i)^{1-\tau_t}\ell(di)\\
& \qquad - (\int_i a_{t-1}^i\ell(di) - B_{t-1})^\alpha(\int_i y_t^i l_t^i\ell(di))^{1-\alpha} - B_t\Big). && (166)
\end{aligned}$$

where:

$$c_t^i = -a_t^i + R_t a_{t-1}^i + w_t (y_t^i l_t^i)^{1-\tau_t} \tag{167}$$

FOC with respect to savings choices. Deriving (165) with respect to a_t^i yields:

$$\begin{aligned}
0 =& \beta^t \int_j \omega_t^j U_c(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \ell(dj) \\
&- \beta^t \int_j \left(\lambda_{c,t}^j - R_t \lambda_{c,t-1}^j \right) U_{cc}(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \\
&+ \beta^t \int_j \lambda_{l,t}^j U_{cl}(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \ell(dj) \\
&+ \beta^t (1-\tau_t) w_t \int_j \lambda_{l,t}^j (y_t^j)^{1-\tau_t} (l_t^j)^{-\tau_t} U_{cc}(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \ell(dj) \\
&+ \beta^{t+1} \mathbb{E}_t \left[\int_j \omega_{t+1}^j U_c(c_{t+1}^j, l_{t+1}^j) \frac{\partial c_{t+1}^j}{\partial a_t^i} \right] \\
&- \beta^{t+1} \mathbb{E}_t \left[\int_j \left(\lambda_{c,t+1}^j - R_{t+1} \lambda_{c,t}^j \right) U_{cc}(c_{t+1}^j, l_{t+1}^j) \frac{\partial c_{t+1}^j}{\partial a_t^i} \ell(dj) \right] \\
&+ \beta^t \mathbb{E}_t \left[\int_j \lambda_{l,t+1}^j U_{cl}(c_{t+1}^i, l_{t+1}^i) \frac{\partial c_{t+1}^j}{\partial a_t^i} \ell(dj) \right] \\
&+ \beta^{t+1} (1-\tau_{t+1}) w_{t+1} \mathbb{E}_t \left[\int_j \lambda_{l,t+1}^j (y_{t+1}^j)^{1-\tau_{t+1}} (l_{t+1}^j)^{-\tau_{t+1}} U_{cc}(c_{t+1}^j, l_{t+1}^j) \frac{\partial c_{t+1}^j}{\partial a_t^i} \ell(dj) \right] \\
&+ \beta^{t+1} \mathbb{E}_t \left[\mu_{t+1} \left(\alpha K_t^{\alpha-1} L_{t+1}^{1-\alpha} - (r_{t+1} + \delta) \right) \right]
\end{aligned}$$

We also denote:

$$\begin{aligned}
\psi_t^i =& \omega_t^i U_c(c_t^i, l_t^i) + \lambda_{l,t}^i U_{cl}(c_t^i, l_t^i) \\
&- \left(\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i - \lambda_{l,t}^i (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t} \right) U_{cc}(c_t^i, l_t^i).
\end{aligned} \tag{168}$$

and get using $\tilde{r}_{t+1} = \alpha K_t^{\alpha-1} L_{t+1}^{1-\alpha} - \delta$:

$$\begin{aligned}
0 =& \int_j \psi_t^j \frac{\partial c_t^j}{\partial a_t^i} \ell(dj) + \beta \mathbb{E}_t \left[\int_j \psi_{t+1}^j \frac{\partial c_{t+1}^j}{\partial a_t^i} \right] \\
&+ \beta \mathbb{E}_t \left[\mu_{t+1} (\tilde{r}_{t+1} - R_{t+1} + 1) \right].
\end{aligned}$$

Using (167), we obtain $\frac{\partial c_t^j}{\partial a_t^i} = -1_{i=j}$ and $\frac{\partial c_{t+1}^j}{\partial a_t^i} = R_{t+1} 1_{i=j}$, from which we deduce:

$$\psi_t^i = \beta \mathbb{E}_t \left[R_{t+1} \psi_{t+1}^i \right] + \beta \mathbb{E}_t \left[\mu_{t+1} (1 + \tilde{r}_{t+1} - R_{t+1}) \right].$$

FOC with respect to labor supply. Deriving (165) with respect to l_t^i yields:

$$\begin{aligned}0 = &\int_j \psi_t^j \frac{\partial c_t^j}{\partial l_t^i} \ell(dj) - \psi_{l,t}^j \\ &- \lambda_{l,t}^i (1-\tau_t)\tau_t w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t - 1} U_c(c_t^i, l_t^i) - \mu_t (w_t(1-\tau_t)(y_t^i)^{1-\tau_t}(l_t^i)^{-\tau_t} - F_{L,t} y_t^i),\end{aligned}$$

where we have defined:

$$\psi_{l,t}^i = -\omega_t^i U_l(c_t^i, l_t^i) - \lambda_{l,t}^i U_{ll}(c_t^i, l_t^i) + (\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i - \lambda_{l,t}^i (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t}) U_{cl}(c_t^i, l_t^i).$$

Using (167), we obtain $\frac{\partial c_t^j}{\partial l_t^i} = (1-\tau_t) w_t (y_t^j)^{1-\tau_t} (l_t^i)^{-\tau_t} 1_{i=j}$,which implies:

$$\begin{aligned}\psi_{l,t}^i = &(1-\tau_t) w_t y_t^i (y_t^i l_t^i)^{-\tau_t} \hat{\psi}_t^i \\ &+ \mu_t F_{L,t} y_t^i - \lambda_{l,t}^i (1-\tau_t)\tau_t w_t y_t^i (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i)/l_t^i.\end{aligned}$$

FOC with respect to the interest rate. Deriving (165) with respect to R_t yields:

$$0 = \int_j \left(\psi_t^j \frac{\partial c_t^j}{\partial R_t} + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j) \right) \ell(dj) - \mu_t \int_j a_{t-1}^j \ell(dj).$$

From (167), we obtain $\frac{\partial c_t^j}{\partial R_t} = a_{t-1}^j$, which yields:

$$0 = \int_j \left(\hat{\psi}_t^j a_{t-1}^j + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j) \right) \ell(dj).$$

FOC with respect to the wage rate. Deriving (165) with respect to w_t yields:

$$\begin{aligned}0 = &\int_j \left(\psi_t^j \frac{\partial c_t^j}{\partial w_t} + \lambda_{l,t}^j (1-\tau_t)(y_t^j)^{1-\tau_t} (l_t^j)^{-\tau_t} U_c(c_t^j, l_t^j) \right) \ell(dj) \\ &- \mu_t \int_j (y_t^j l_t^j)^{1-\tau_t} \ell(dj)\end{aligned}$$

From (167), we get $\frac{\partial c_t^j}{\partial w_t} = (y_t^j l_t^j)^{1-\tau_t}$ and:

$$0 = \int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j \right) \ell(dj).$$

FOC with respect to public debt. Deriving (165) with respect to B_t yields:

$$0 = \mu_t - \beta \left[(1 - \delta - \alpha K_{t-1}^{\alpha} L_t^{1-\alpha} \mu_{t+1} \right],$$

or using the definition of $\tilde{r}_{t+1}$:

$$\mu_t = \beta(1 + \tilde{r}_{t+1})\mu_{t+1}.$$

FOC with respect to progressivity. Deriving (165) with respect to τ_t yields:

$$\begin{aligned}
0 = & \int_j \psi_t^j \frac{\partial c_t^j}{\partial \tau_t} \ell(dj) \\
& + w_t \int_j \lambda_{l,t}^j \frac{\partial}{\partial \tau_t} \left((1-\tau_t)(y_t^j l_t^j)^{1-\tau_t} \right) (U_c(c_t^j, l_t^j)/l_t^j)\ell(dj) \\
& - \mu_t w_t \int_j \frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t} \right) \ell(dj).
\end{aligned}$$

From (167), we have $\frac{\partial c_t^j}{\partial \tau_t} = (y_t^j l_t^j)^{1-\tau_t}$ and:

$$\begin{aligned}
0 = & \int_j \hat{\psi}_t^j \frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t} \right) (dj) \\
& + \int_j \lambda_{l,t}^j \left(-(y_t^j l_t^j)^{1-\tau_t} + (1-\tau_t) \frac{\partial}{\partial \tau_t} (y_t^j l_t^j)^{1-\tau_t} \right) (U_c(c_t^j, l_t^j)/l_t^j)\ell(dj).
\end{aligned}$$

and

$$\begin{aligned}
0 = & \int_j \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j \right) \frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t} \right) (dj) \\
& - \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t} (U_c(c_t^j, l_t^j)/l_t^j)\ell(dj).
\end{aligned}$$

Using $\frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t} \right) = -\ln(y_t^j l_t^j)(y_t^j l_t^j)^{1-\tau_t}$, we finally deduce:

$$\begin{aligned}
0 = & \int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j \right) \ln(y_t^j l_t^j)(dj) \\
& + \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t} (U_c(c_t^j, l_t^j)/l_t^j)\ell(dj).
\end{aligned}$$

Summary of FOCs.

$$
\begin{aligned}
\hat{\psi}_t^i &= \beta \mathbb{E}_t \left[(1+r_{t+1}) \hat{\psi}_{t+1}^i \right], \\
\psi_{l,t}^i &= (1-\tau_t) w_t y_t^i (y_t^i l_t^i)^{-\tau_t} \hat{\psi}_t^i \\
&\quad + \mu_t F_{L,t} y_t^i - \lambda_{l,t}^i (1-\tau_t) \tau_t w_t y_t^i (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i)/l_t^i, \\
\mu_t &= \beta (1+\tilde{r}_{t+1}) \mu_{t+1} \\
0 &= \int_j \left(\hat{\psi}_t^j a_{t-1}^j + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j) \right) \ell(dj), \\
0 &= \int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j \right) \ell(dj), \\
0 &= \int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j \right) \ln(y_t^j l_t^j)(dj) \\
&\quad + \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t} (U_c(c_t^j, l_t^j)/l_t^j) \ell(dj).
\end{aligned}
$$

B.1 Checking that FOCs are identical

We check here that the first-order conditions of the Ramsey program derived in the general case of Section 2.6 exactly simplify to the first-order conditions derived in the specific case of Section 3. We start with expressing ψ_t^i and $\psi_{l,t}^i$ (equations (55) and (59)) in the context of the GHH utility function. We denote by $C = c - \chi^{-1} \frac{l^{1+1/\varphi}}{1+1/\varphi}$. Since $U(c,l) = \ln \left(c - \chi^{-1} \frac{l^{1+1/\varphi}}{1+1/\varphi} \right)$, we compute:

$$
\begin{aligned}
U_c(c,l) &= \frac{1}{C},\ U_{cc}(c,l) = -\frac{1}{C^2},\ U_l(c,l) = -\chi^{-1} l^{1/\varphi} \frac{1}{C}, \\
U_{ll}(c,l) &= -\frac{\chi^{-1} l^{1/\varphi - 1}}{C} \left(\frac{1}{\varphi} + \frac{\chi^{-1} l^{1/\varphi}}{C} \right),\ U_{cl}(c,l) = \frac{\chi^{-1} l^{1/\varphi}}{C^2}.
\end{aligned}
$$

Plugging this into equations (55) and (59) and using the labor Euler equation (11) stating that $\chi^{-1} l_t^{i,1/\varphi} = y_t^i w_t$, we deduce that the expressions of ψ_t^i and $\psi_{l,t}^i$ become:

$$
\psi_t^i C_t^i = 1 + \left(\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i \right) \frac{1}{C_t^i}, \tag{169}
$$

$$
\psi_{l,t}^i C_t^i = y_t^i w_t \left(1 + \frac{\lambda_{l,t}^i}{\varphi l_t^i} + (\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i) \frac{1}{C_t^i} \right). \tag{170}
$$

We now turn to the FOCs. Note that FOC (60) is exactly the same as FOC (31), while FOC (63) has no equivalent in the simplified version since the progressivity parameter τ_t is set to zero. FOC (58) can also be written with $\tau_t = 0$: $\psi_{l,t}^i = w_t y_t^i \psi_t^i + \mu_t (F_{L,t} - w_t) y_t^i$. Plugging (169) and (170) yields:

$$
\frac{\lambda_{l,t}^i}{\varphi l_t^i} \frac{y_t^i w_t}{C_t^i} = \mu_t (F_{L,t} - w_t) y_t^i,
$$

which is equivalent to $0 = 0$ for unemployed agents since their productivity is null. For employed agent with a productivity normalized to one, it becomes:

$$\lambda_{e,l,t} = \varphi \mu_t l_{e,t} C_{e,t} \frac{\tau_t^L}{1-\tau_t^L}. \tag{171}$$

The three remaining FOCs are equations (57), (61), and (62). Taking advantage of the deterministic transitions between employment and unemployment, as well as the fact that unemployed agents are credit-constrained (implying $a_{u,t-1} = \lambda_{u,c,t-1} = 0$) with null productivity, these three FOCs can also be written as follows ($a_{e,t-1}, l_{e,t} > 0$):

$$\psi_{e,t} - \mu_t = \beta R_{t+1}(\psi_{u,t+1} - \mu_{t+1}), \tag{172}$$

$$\mu_t C_{u,t} = \psi_{u,t} C_{u,t} + \frac{\lambda_{e,c,t-1}}{a_{e,t-1}}, \tag{173}$$

$$\mu_t C_{e,t} = \psi_{e,t} C_{e,t} + \frac{\lambda_{e,l,t}}{l_{e,t}}, \tag{174}$$

while similarly expressions of ψ_t^i in (169) can further be specified as:

$$\psi_{e,t} C_{e,t} = 1 + \frac{\lambda_{e,c,t}}{C_{e,t}}, \tag{175}$$

$$\psi_{u,t} C_{u,t} = 1 - R_t \lambda_{e,c,t-1} \frac{1}{C_{u,t}}. \tag{176}$$

Combining (173) and (176) with $a_{e,t-1} = \frac{C_{u,t}}{R_t}$ (which is unemployed agents' budget constraint (17)) implies:

$$\mu_t C_{u,t} = 1, \tag{177}$$

with the expression of $C_{u,t} = R_t \frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi}$ is identical to FOC (32).

Using the consumption Euler equation (21) stating that $\frac{1}{C_{e,t}} = \beta R_{t+1} \frac{1}{C_{u,t+1}}$, the budget constraints (16) and (17) implying that $C_{u,t} = \beta R_t C_{e,t-1}$, and (177) meaning that $1 = \beta \mu_{t+1} R_{t+1} C_{e,t}$, we deduce from (172) and (175):

$$\frac{\lambda_{e,c,t}}{C_{e,t}} = \frac{\beta}{1+\beta}(\mu_t C_{e,t} - 1). \tag{178}$$

Finally, we turn to FOC (174). Combined with the expressions of $\lambda_{e,l,t}$ in (171), $\psi_{e,t}$ in (175), and of $\lambda_{e,c,t}$ in (178), it becomes:

$$C_{e,t}\mu_t \left(1 - (1+\beta)\varphi \frac{\tau_t^L}{1-\tau_t^L}\right) = 1. \tag{179}$$

Using the budget constraint (16) stating that $C_{e,t} = \frac{w_t(\chi w_t)^{\varphi}}{(1+\beta)(1+\varphi)}$, equation (179) becomes FOC (30). This completes the proof that the generic FOCs of Section 2.6 exactly im

C The Ramsey program on the truncated model

C.1 Formulation

We define the set of $(\xi^{u,0}_{y^N})_{y^N}$ such that:

$$\sum_{y^t\in\mathcal{Y}^t \big| (y^t_{t-N+1},\ldots,y^t_t)=y^N} u(c_t(y^t)) = \xi^{u,0}_{y^N} u\Bigg(\sum_{y^t\in\mathcal{Y}^t \big| (y^t_{t-N+1},\ldots,y^t_t)=y^N} c_t(y^t)\Bigg),$$

or compactly:

$$\xi^{u,0}_{y^N} u(c_{t,y^N}) := \sum_{y^N} u(c^i_t).$$

Similarly, we define $(\xi^{v,0}_{y^N})$, $(\xi^{u,1}_{y^N})$, $(\xi^{\tau}_{y^N})$, and $(\xi^{v,1}_{y^N})$ such that:

$$\begin{aligned}
\xi^{v,0}_{y^N} v(l_{t,y^N}) &:= \sum_{y^N} v(l^i_t), \\
\xi^{u,1}_{y^N} u'(c_{t,y^N}) &:= \sum_{y^N} u'(c^i_t), \\
\xi^{\tau}_{y^N} \sum_{y^N} (l_{t,y^N})^{\tau_t} &:= \sum_{y^N} (l^i_t)^{\tau_t}, \\
\xi^{v,1}_{t,s^N} v'(l_{t,y^N}) &:= \tau_t w_t \xi^{\tau}_{y^N} (l_{t,y^N} y_{y^N})^{1-\tau_t} \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}).
\end{aligned}$$

The Ramsey problem can then be written as:

$$\max_{\left(r_t,\tilde{w}_t,\tilde{r}_t,\tau^K_t,\tau_t,\kappa_t,B_t,K_t,L_t,(a^i_t,c^i_t,l^i_t,\nu^i_t)_i\right)_{t\geq 0}} \mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \sum_{y^N} S_{t,y^N} \omega^i_t (\xi^{u,0}_{y^N} u(c_{t,y^N}) - \xi^{v,0}_{y^N} v(l_{t,y^N}))\right],$$

$$G_t + T_t + (1+r_t)B_{t-1} + r_t K_{t-1} + w_t \xi^{y}_{y^N} \sum_{y^N} (l_{t,y^N} y_{y^N})^{1-\tau_t} \ell(di) = F(K_{t-1}, L_t, z_t) + B_t,$$

for all $y^N \in \mathcal{Y}$: $, c_{t,y^N} + a_{t,y^N} = w_t \xi^{y}_{y^N} (l_{t,y^N} y_{y^N})^{1-\tau_t} + (1+r_t)\tilde{a}_{t,y^N} + T_t,$

$$\begin{aligned}
& a_{t,y^N} \geq -\bar{a},\ \nu_{t,y^N}(a_{t,y^N} + \bar{a}) = 0,\ \nu_{t,y^N} \geq 0, \\
& \xi^{u,E}_{y^N} u'(c_{t,y^N}) = \beta \mathbb{E}_t\Bigg[\sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{t,y^N \tilde{y}^N} \xi^{u,E}_{\tilde{y}^N} u'(c_{t+1,\tilde{y}^N})(1+r_{t+1})\Bigg] + \nu_{t,y^N}, \\
& \xi^{v,1}_{t,s^N} v'(l_{t,y^N}) \equiv \tau_t w_t \xi^{y}_{y^N} (l_{t,y^N} y_{y^N})^{1-\tau_t} \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}), \\
& K_t + B_t = \sum_{y^N} S_{t,y^N} a_{t,y^N},\ L_t = \sum_{y^N} S_{t,y^N} y^i_{t,y^N} l_{t,y^N}.
\end{aligned}$$

C.2 Factorization

We now factorize the Ramsey problem of Section C.1. We define:

$$
\begin{aligned}
J =&\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{y^N \in \mathcal{Y}} \Big[S_{t,y^N} \Big(\Big(\omega_{y^N} \xi_{y^N}^{u,0} u(c_{t,y^N}) - \xi_{y^N}^{v,0} v(l_{t,y^N}) \Big) \\
&- \Big(\lambda_{c,t,y^N} - \tilde{\lambda}_{c,t,y^N}(1+r_t) \Big) \xi_{y^N}^{u,1} U_c(c_{t,y^N}, l_{t,y^N}) \Big), \\
&- \lambda_{l,t,y^N} \left(v'(l_{t,y^N}) - \tau_t w_t (y_{t,y^N})^{\tau_t} \xi_{y^N}^{y} (l_{t,y^N})^{-\tau_t} \xi_{y^N}^{u,1} u'(c_{t,y^N}) \right) \Big].
\end{aligned}
$$

The Ramsey program becomes maximizing J subject to the following constraints:

$$
\begin{aligned}
&G_t + T_t + (1+r_t)B_{t-1} + r_t K_{t-1} + w_t \xi_{y^N}^{\tau} \sum_{y^N} (l_{t,y^N} y_{y^N})^{1-\tau_t} \ell(di) = F(K_{t-1}, L_t, z_t) + B_t \\
&\text{for all } y^N \in \mathcal{Y}:\ , c_{t,y^N} + a_{t,y^N} = w_t \xi_{y^N}^{y} (l_{t,y^N} y_{y^N})^{1-\tau_t} + (1+r_t)\tilde{a}_{t,y^N} + T_t, \\
&\qquad a_{t,y^N} \geq -\bar{a},\ \nu_{t,y^N}(a_{t,y^N} + \bar{a}) = 0,\ \nu_{t,y^N} \geq 0, \\
&\qquad K_t + B_t = \sum_{y^N} S_{t,y^N} a_{t,y^N},\ L_t = \sum_{y^N} S_{t,y^N} y_{t,y^N}^{i} l_{t,y^N}.
\end{aligned}
$$

C.3 FOCs of the planner

Before expressing the FOCs of the Ramsey program, we define:

$$
\begin{aligned}
\hat{\psi}_{t,y^N} :=& \omega_{y^N} \xi_{y^N}^{u,0} u'(c_{t,y^N}) - \mu_t \\
&- \left(\lambda_{c,t,y^N} \xi_{y^N}^{u,E} - (1+r_t)\tilde{\lambda}_{c,t,y^N} \xi_{y^N}^{u,E} - \lambda_{l,t,y^N} \xi_{y^N}^{y} \tau_t w_t (y_0^N)^{1-\tau_t} l_{t,y^N}^{-\tau_t} \xi_{y^N}^{u,1} \right) u''(c_{t,y^N}).
\end{aligned}
$$

The two Euler equations can be written as follows:

$$
\begin{aligned}
&\xi_{y^N}^{u,E} u'(c_{t,y^N}) = \beta \mathbb{E}_t \Big[\sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{t,y^N \tilde{y}^N} \xi_{\tilde{y}^N}^{u,E} u'(c_{t+1,\tilde{y}^N})(1+r_{t+1}) \Big] + \nu_{t,y^N}, \\
&\xi_{t,s^N}^{v,1} v'(l_{t,y^N}) = (1-\tau_t)\, w_t \xi_{y^N}^{\tau} (l_{t,y^N} y_{y^N})^{1-\tau_t} \xi_{y^N}^{u,1} (u'(c_{t,y^N})/l_{t,y^N})
\end{aligned}
$$

while the constraints of the Ramsey program become:

$$
\begin{aligned}
B_t + K_{t-1}^{\alpha} L_t^{1-\alpha} &= G_t + T_t + (1-\delta)B_{t-1} + (r_t+\delta)A_{t-1} \\
&\quad + w_t \sum_{y^N \in \mathcal{Y}^N} S_{t,y^N} \xi_{y^N}^{\tau} (y_{y^N} l_{t,y^N})^{1-\tau_t}, \\
\tilde{\lambda}_{t,y^N} &= \frac{1}{S_{t,y^N}} \sum_{\tilde{y}^N \in \mathcal{Y}^N} S_{t-1,\tilde{y}^N} \lambda_{t-1,\tilde{y}^N} \Pi_{t,\tilde{y}^N,y^N}, \\
c_{t,y^N} + a_{t,y^N} &= w_t (l_{t,y^N} y_{y^N})^{1-\tau_t} + (1+r_t)\tilde{a}_{t,y^N} + T_t, \\
a_{t,y^N} \geq 0 \text{ and } \tilde{a}_{t,y^N} &= \sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{\tilde{y}^N y^N,t} \frac{S_{t-1,\tilde{y}^N}}{S_{t,y^N}} a_{t-1,\tilde{y}^N}.
\end{aligned}
$$

The FOCs of the Ramsey program can finally be written as follows:

$$
\begin{aligned}
\hat{\psi}_{t,y^N} &= \beta \mathbb{E}_t \Big[(1+r_{t+1}) \sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{t,y^N \tilde{y}^N} \hat{\psi}_{t+1,\tilde{y}^N} \Big] \text{ if } \nu_{y^N} = 0 \text{ and } \lambda_{t,y^N} = 0 \text{ otherwise}, \\
\hat{\psi}_{t,y^N} &= \frac{1}{\tau_t w_t \xi_{y^N}^{\tau} (y_0^N)^{1-\tau_t} l_{t,y^N}^{\tau_t}} (\omega_{y^N} \xi_{y^N}^{v,0} v'(l_{t,y^N}) + \lambda_{l,t,y^N} \xi_{y^N}^{v,1} v''(l_{t,y^N})) \\
&\quad + \lambda_{l,t,y^N} \tau_t \xi_{y^N}^{u,1} (u'(c_{t,y^N})/l_{t,y^N}) \\
&\quad - \mu_t (1-\alpha) \frac{Y_t}{\tau_t w_t \xi_{y^N}^{\tau} (y_0^N)^{-\tau_t} l_{t,y^N}^{-\tau_t} L_t}, \\
0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \left(\hat{\psi}_{t,y^N} \tilde{a}_{t,y^N} + \tilde{\lambda}_{c,t,y^N} \xi_{y^N}^{u,E} u'(c_{t,y^N}) \right), \\
0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \xi_{y^N}^{\tau} (l_{t,y^N} y_{y^N})^{\tau_t} \left(\hat{\psi}_{t,y^N} + \lambda_{l,t,y^N} (1-\tau_t) \xi_{y^N}^{u,1} (u'(c_{t,y^N})/l_{t,y^N}) \right), \\
0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \hat{\psi}_{t,y^N}, \\
\mu_t &= \beta \mathbb{E} \left[\mu_{t+1} \left(1 + \alpha \frac{Y_{t+1}}{K_t} - \delta \right) \right], \\
0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \lambda_{l,t,y^N} \xi_{y^N}^{\tau} (l_{t,y^N} y_{y^N})^{1-\tau_t} \xi_{y^N}^{u,1} (u'(c_{t,y^N})/l_{t,y^N}) \\
&\quad + \sum_{y^N \in \mathcal{Y}} S_{y^N} \left(\hat{\psi}_{t,y^N} + \lambda_{l,t,y^N} (1-\tau_t) \xi_{y^N}^{u,1} (u'(c_{t,y^N})/l_{t,y^N}) \right) \ln \left(l_{t,y^N} y_{y^N} \right) \xi_{y^N}^{\tau} (l_{t,y^N} y_{y^N})^{1-\tau_t}.
\end{aligned}
$$